STUDY MATERIAL

STRENGTH OF MATERIALS

(FOR DIPLOMA & POLYTECHNIC STUDENTS)

3RD SEMESTER

INTRODUCTION

While selecting a suitable material, for his project, an engineer is always interested to know its strength.

The strength of a material may be defined as *ability to resist its failure & behavior, under the action of external forces.*

It has been observed that, under the action of these forces, the material is first deformed & then its failure takes place. A detailed study of forces & their effects, along with some suitable protective measures for the safe working conditions is known as ' strength of materials '.

- **TENSILE STRENGTH : (σ_{uts} or S_u)**

  It is the stress at which a material breaks or permanently deforms. It is an intensive property & consequently doesn't depend on the size of the test specimen. However it is dependent on the preparation of the specimen & the temperature of the test environment & material.
- **YIELD STRENGTH :**

  The stress at which material strain changes from elastic deformation to plastic deformation, causing it to deform permanently.
- **ULTIMATE STRENGTH :**

  The maximum stress a material can withstand when subjected to tension, compression or shearing. It is the maximum stress on the stress-strain curve.
- **BREAKING STRENGTH :**

  The stress co-ordinates on the stress-strain curve at the point of rupture.
- **COMPRESSIVE STRENGTH :**

  It is the capacity of a material to withstand axially directed pushing forces. When the limit of compressive strength is reached, materials are crushed.
- **SHEAR STRENGTH :**

  The strength of a material or component against the type of yield or structural failure where the material or component fails in shear.

SIMPLE STRESS & STRAIN

- **DEFORMATION :**

When an external force acts on a body, the body tends to undergo some change in dimension (change in shape & size).This changes in dimension is known as ' **deformation** '. The amount of deformation depends upon the amount of force or load and the nature of material. It has the unit same as dimension. This can be result of tensile (pulling) force, compressive (pushing) force, shear, bending or torsion (twisting).

- **TYPES OF DEFORMATION :**

Depending on the type of material, size & geometry of object & the forces applied, various types of deformation may occur.

a) **ELASTIC DEFORMATION :** On removal of load, if the body regains its original shape & size, then the body is said to be in elastic state & the deformation is said to be ' elastic deformation '. This type of deformation is reversible .

   Soft thermo plastics & metals have moderate elastic deformation ranges, while ceramics, crystals & hard thermosetting plastics undergo almost no elastic deformation. Elastic deformation governed by **" HOOKE'S LAW '.**

b) **PLASTIC DEFORMATION :** On removal of load, if the body is not regain its original shape & size, then the body is said to be in plastic state & this deformation is known as ' plastic deformation '. This type of deformation is irreversible.

   One material with a large plastic deformation range is wet chewing gum.

Due to cohesive force between the molecules of material, the body resists the deformation, the resistance/unit area is called as stress.

- ***STRESS :-***

As the body undergoes deformation, it sets up some resistance to deformation. This resistance (opposing force) per unit area to deformation is known as 'stress'.

Stress may be defined as the resisting force per unit area.

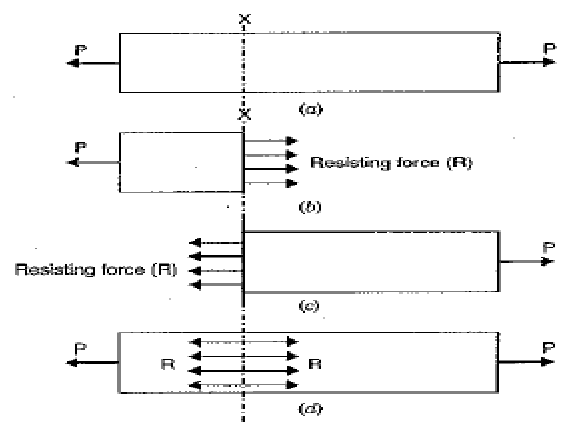
Mathematically, It can be written as –

$$\sigma \text{ or } f = \frac{\textit{Resisting Force } (R)}{\textit{Cross-sectional area } (A)} = P/A$$

Where R = P =load acting on body in N or KN

A = cross-sectional area of body in mm^2

σ or f = stress in N/mm^2



- ***UNITS OF STRESS***

In C.G.S. ---- **gmf/cm²** or **dyne/cm²**

In M.K.S. --- **kgf/cm²** or **kg/cm²**

In S.I. ------- **N /m², N / mm²**, Pascal , (**1 pascal = 1 N / m²**),

1 MPa = 10^6 Pa = 1 N / mm²,

1 GPa = 10^9 Pa = 10^3 N / mm² = 1kN mm².

- **STRAIN :-**

It can be defined as the ratio of change in dimension of a body (when subjected to a external load) to the original dimension. Strain is a unit less quantity.

Strain may be defined as the ratio of change in length to the original length & can be written as –

ε or e = δl / l

Where, l = original length in mm

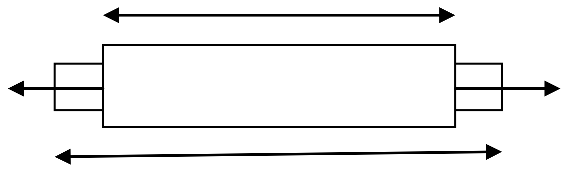
δl = change in length in mm

TYPES OF STRESSES : There are 3 types of simple stresses –

a) Tensile stress
b) Compressive stress
c) Shear stress

a) TENSILE STRESS :-

When a section is subjected to two equal & opposite pulls, as a result of which the body tends to lengthen (increase in length) & the stress induced is called as 'tensile stress'.



σ_t **or** f_t **= P / A**

where, f_t = tensile stress

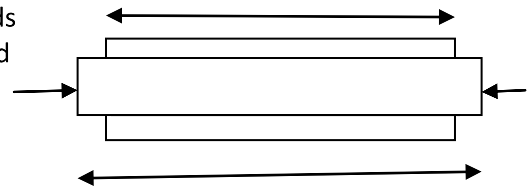
P = pull

A = cross sectional area of the section

δl = increase in length (elongation)

b) COMPRESSIVE STRESS :-

When a section is subjected to two equal and Opposite pushes, as a result of which the body tends To shorten (decrease in length) & the stress induced is called 'compressive stress'.



σ_c **or** f_c **= P / A**

Where, f_c = compressive stress

P = push

A = cross sectional area of the section

δl= decrease in length (contraction

c) SHEAR STRESS :-

When a section is subjected to two equal & opposite forces, acting tangentially across the resisting section, as a result of which the body tends to shear off across the section & the stress induced is called shear stress.

τ or f_s **= Q / A** where, τ or f_s = shear stress

Q = Shear force acting tangentially across the section

A = area of surface on which the force acts

- ***TYPES OF STRAINS :-*** There are 4 types of strains –

a) Tensile strain

b) Compressive strain

c) Shear strain

d) Volumetric strain

a) TENSILE STRAIN :-

When a body subjected to tensile stress, which means increase in length, then the ratio of increase in length to the original length is called 'tensile strain'.

ε or *e* = δl / l

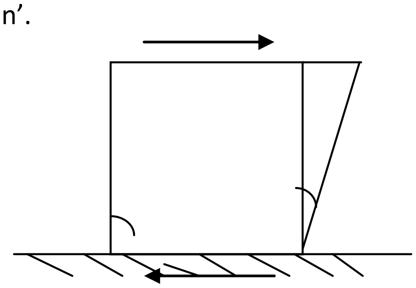
b) COMPRESSIVE STRAIN :-

When a body is subjected to compressive stress, which means decrease in length, then the ratio of decrease in length to the original length is called 'compressive strain'.

ε or *e* = δl / l

c) SHEAR STRAIN :-

When a body is subjected to shear stress which means the angular deformation, then this angle is known as 'shear strain'.



Consider a cube of length of 'l' fixed at the bottom face AB. Let a force 'Q' be applied at the face 'DC' tangentially. As a result of the force 'Q' let the cube be distorted from ABCD to ABC'D' through an angle 'Ø' as shown in fig.

We know that, shear strain = deformation / Original length

= CC' / BC = tan Ø

Shear strain = Ø (when Ø is very small, tan Ø = Ø)

d) VOLUMETRIC STRAIN :-

Volumetric strain is the ratio of change in volume to the original volume.

Mathematically, e_v **= δv / v**

Where, v = original volume

δv = change in volume

e_v = volumetric strain

- When a rectangular block subjected to axial load,

  its volumetric strain,

  $e_v = \delta v / v = \delta l / l - \delta b / b - \delta t / t = e - e.\mu - e.\mu = e(1 - 2\mu)$

- When a circular sectional bar is subjected to axial load,

  its volumetric strain,

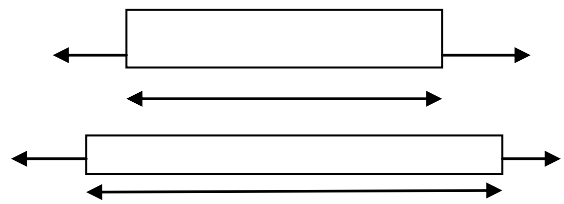
  $e_v = \delta v / v = \delta l / l - \delta d / d - \delta d / d = e - e.\mu - e.\mu = e(1 - 2\mu)$

  $e_v = \delta v / v = e(1 - 2\mu)$

  where, e = linear strain & μ = Poisson's ratio.

- **LINEAR STRAIN & LATERAL STRAIN :-**

Consider linear deformation of a circular bar of length 'l' and diameter 'd' subjected to a tensile force 'P'. The bar will extend through a length 'δl' which will be followed by the decrease in diameter from 'd' to 'd – δd' it is thus obvious that every stress is always accompanied by a strain in its own direction known as ' linear strain ' & an opposite kind of strain in every direction at right angles to it. Such strain is known as 'lateral strain'.



Linear strain = $\delta l / l$

Lateral strain = $\delta d / d$

Where, δl = change in length &

δd = change in diameter

LONGITUDINAL or LINEAR STRAIN :

It is the ratio of the axial deformation due to axial load to its original length.

Longitudinal strain = $\delta l / l$

Where, δl = Longitudinal deformation or change in length,

l = Original length.

- **LATERAL STRAIN :**

  It is the lateral (or transverse) deformation per unit depth or breadth of a rectangular block. (or per unit dia. In case of circular member)

  Lateral strain = $\delta b / b$ or $\delta t / t$ - in case of rectangular section subjected to axial load.

  = $\delta d / d$ - in case of circular section subjected to axial load.

  Where, δb, δt & δd are lateral deformation in breadth, thickness & dia. respectively.

  b, t & d are original dimensions.

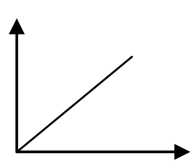
- **ELASTIC LIMIT :-** A material is said to be elastic, if it regains its original shape & size on the removal of external load acting on the body.

It has been found that for a given section, there is a limiting value of force up to which, the deformation entirely disappears on removal of external load. The value of stress corresponding to this limiting force is called 'elastic limit'. If a material is stretched beyond elastic limit, it cannot regain its shape & size entirely on the removal of external load. This stage is said to be 'plastic stage'.

- **HOOKE'S LAW OF ELASTICITY :- (Robert Hooke - 1678)**

Hooke's law states that, 'Within elastic limit, the stress is directly proportional to strain'

In other words, Within elastic limit, the ratio of stress to corresponding strain is a constant which is characteristic of material.



Mathematically, $f \propto e$

i.e. $f = E . e$

i.e. $f / e = E$

Thus, stress / strain = E (a constant ,Elastic constants)

It may be noted that Hooke's law holds good for tension as well as compression.

- **ELASTIC CONSTANTS :-**

There are 3 types of elastic constants –

a. **YOUNG'S MODULUS or MODULUS OF ELASTICITY –(E)**

It is defined as the ratio of linear stress (tensile or compressive) to the corresponding strain. This is denoted by ' E '.

Mathematically, Young's modulus, **E = linear stress / linear strain** $= \sigma / e = \dfrac{(P/A)}{(\delta l / l)} = \dfrac{P.l}{A.\delta l}$

b. SHEAR MODULUS or MODULUS OF RIGIDITY :- (C, N or G)

It is defined as the ratio of shear stress to the corresponding shear strain. It is denoted by C, N or G.

Mathematically, Modulus of Rigidity, N = Shear stress / Shear strain = τ / ϕ

c. BULK MODULUS :- (K)

It is defined as the ratio of direct stress to the volumetric strain, when a body is subjected to stresses in more than one direction. It is denoted by 'K'.

Mathematically, **Bulk Modulus, K = Direct stress / volumetric strain = σ / e_v =** $\dfrac{\sigma}{(\delta v / v)}$

- **DEFORMATION OF A BODY DUE TO FORCE ACTING ON IT :**

Consider a body subjected to a tensile stress.

Let P = Load or force acting on the body.

l = Length of the body

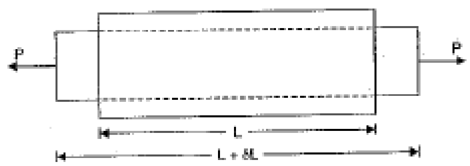
A = Cross-sectional area of the body

σ = Stress induced in the body

E = Modulus of elasticity for the material of the body

ε = strain &

δl = Deformation of the body



We know that the stress, **σ = P / A & strain ε = σ / E = P / AE**

And deformation , **$\delta l = \varepsilon . l = \dfrac{P.l}{A.E}$**

- **DEFORMATION OF A BODY DUE TO SELF WEIGHT :**

Consider a bar AB hanging freely under its own weight.

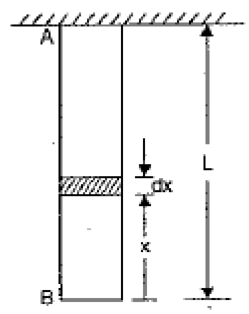
Let l = Length of the bar

A = Cross-sectional area of the bar

E = Young's modulus for the bar material

& w = Specific weight of the bar material

Now consider a small section *dx* of the bar at a distance x from B. We know that the weight of the bar for a length of x **, P = wAx**



Elongation of the Small Section of the bar, due to weight of bar for small section x,

= Pl / AE = (wAx.dx) / AE = wx.dx /E

Total elongation may be found out by integrating the above equation between 0 & l,

$$\delta l = \int_0^l \frac{wx.dx}{E}$$

$$= \frac{w}{E}\int_0^l x.dx = w/E\left(x^2/2\right) = \frac{wl^2}{2E} = \frac{Wl}{2AE}$$ (W= wAl total weight & wl = W/A)

- **POISSON'S RATIO : (simen poisson)**

It has been experimentally found that if a body is stressed within elastic limit, the lateral strain bears a constant ratio to linear strain. Mathematically,

Lateral strain / linear strain = a constant

This constant is known as 'Poisson's Ratio' & is denoted by 1/m or μ

Poisson's Ratio (1/m or μ) = lateral strain / linear strain = $\dfrac{(\delta d/d)}{(\delta l/l)}$

Lateral strain = **μ** .linear strain = **μ**. E Where, **μ** = Poisson's ratio & ε = linear strain

- **Volumetric strain of a rectangular body subjected to an axial force :**

Vol. strain $\varepsilon_v = \delta V / V = \dfrac{P}{btE}(1- \dfrac{2}{m})$

- **Volumetric strain of a rectangular body subjected to three mutually perpendicular forces :**

Vol. strain $\varepsilon_v = \delta V/V = \boldsymbol{\varepsilon_x + \varepsilon_y + \varepsilon_z}$

Where, $\varepsilon_x = \dfrac{\sigma x}{E} - \dfrac{\sigma y}{mE} - \dfrac{\sigma z}{mE}$ $\varepsilon_y = \dfrac{\sigma y}{E} - \dfrac{\sigma z}{mE} - \dfrac{\sigma x}{mE}$

$$\varepsilon_z = \frac{\sigma z}{E} - \frac{\sigma x}{mE} - \frac{\sigma y}{mE}$$

- **RELATIONSHIP BETWEEN ELASTIC CONSTANTS :-**

a) Relationship between Young's modulus (E) & Modulus of Rigidity (N) --

E = 2N((m+1) /m) i.e. **N = mE / 2(m+1)**

or **E = 2N(1+ μ)** i.e. **N = E / 2(1+ μ)**

b) Relationship between **E & K** -

E = 3K ((m-2) / m) i.e. **K = mE / 3(m-2)**

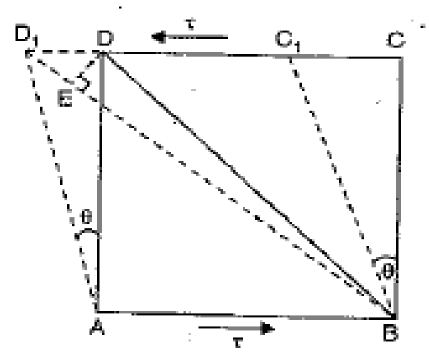
or **E = 3K(1-2 μ)** i.e. **K = E / 3(1-2 μ)**

c) Relation between **E, N & K** -

E = 9KN / (N+3K)

- ***Relation between E & C :***

Consider a cube of length l subjected to a shear stress of τ . Due to these stresses the cube is subjected to some distortion, such that the diagonal BD will be elongated & the diagonal AC will be shortened. Let this shear stress (τ) cause shear strain (ϕ). The diagonal BD is distorted to BD_1.



$$\text{Strain of BD} = \frac{(BD1 - BD)}{BD} = \frac{D1E}{BD}$$

$$= \frac{DD1\cos 45^0}{AD\sqrt{2}} = \frac{DD1}{2AD} = \frac{\varphi}{2}$$

We see that the linear strain of the diagonal BD is half of the shear strain & is tensile in nature. Similarly it can be proved that the linear strain of the diagonal AC is also equal to ϕ/2 but compressive in nature.

Linear strain of diagonal BD = ϕ/2 = τ/2C .... (1)

Where τ = Shear stress & C = Modulus of rigidity.

Now consider the shear stress (τ) acting on the sides AB, CD, CB & AD.

The effect of this stress is to cause tensile stress on BD & compressive stress on AC.

Tensile strain on BD due to tensile stress on BD = $\frac{\tau}{E}$

Tensile strain on BD due to comp. stress on AC = $\mu.\frac{\tau}{E}$

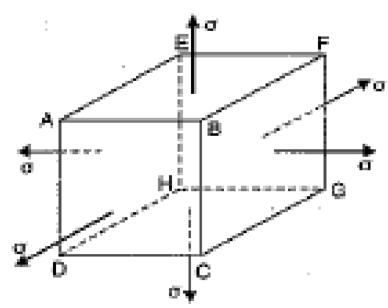
The combined effect of two stresses on BD = $\frac{\tau}{E} + \mu.\frac{\tau}{E} = \frac{\tau}{E}(1+\mu) = \frac{\tau}{E}(1+\mu)$ ....(2)

Equating equation 1 & 2

$$\frac{\tau}{2C} = \frac{\tau}{E}(1+\mu) \quad \text{or} \quad C = \frac{E}{2(1+\mu)} \qquad \text{or} \quad E = 2C(1+\mu)$$

- **Relation between E & K :**

Consider a cube ABCDEFGH as in fig. Let the cube be subjected to 3 mutually perpendicular tensile stresses of equal intensity σ.



Let, σ = Tensile stress on faces of cubes

l = Edge of the cube

δl = change in length of the cube

E = Young's modulus for the material of the cube

μ = Poisson's ratio.

Now consider the deformation of one side of the cube (say AB) under the action of the 3 mutually perpendicular stresses. We know that this side will suffer the following strains due to pair of stresses.

1. Tensile strain equal to σ/E due to stresses on the faces BFCG & AEDH (i.e. σ/E)
2. Compressive lateral strain equal to $\mu.\frac{\sigma}{E}$ due to stresses on faces AEBF & DHCG.($-\mu.\frac{\sigma}{E}$)
3. Compressive lateral strain equal to $\mu.\frac{\sigma}{E}$ due to stresses on faces ABCD & EFGH. .($-\mu.\frac{\sigma}{E}$)

Hence the total strain of AB is given by $\frac{dL}{L} = \frac{\sigma}{E} - \mu.\frac{\sigma}{E} - \mu.\frac{\sigma}{E} = \frac{\sigma}{E}(1-2\mu)$ . . . . (1)

Now original volume of the cube, $V = L^3$ . . . . (2)

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If dL is the change in length, then dV is the change in volume.

Differentiating equation (2), with respect to L, $dV = 3L^2 . dL$ .... (3)

Dividing equation (3) by equation (2), we get

$$\frac{dV}{V} = (3L^2 . dL) / L^3 = \frac{3\,dL}{L}$$

Substituting the value of $\frac{dL}{L}$ from equation (1), in the above equation, we get

$$\frac{dV}{V} = 3\frac{\sigma}{E}(1 - 2\mu)$$

We know that, Bulk Modulus, $K = \frac{\sigma}{\frac{dV}{V}} = \frac{\sigma}{\frac{3\sigma}{E}(1-2\mu)}$ $\left[\frac{dV}{V} = 3\frac{\sigma}{E}(1 - 2\mu)\right]$

$$= \frac{E}{3(1-2\mu)}$$

or $$\boldsymbol{E = 3K(1 - 2\mu)}$$

- **BRITTLE MATERIAL :-**

Brittle material is one which shows very small deformation before it's fracture. (no well defined yield point). Example – Cast iron

- **DUCTILE MATERIAL :-**

Ductile material is one which shows appreciate deformation before fracture. (well defined yield point). Example – Mild steel

- **STRESS - STRAIN CURVE :- For ductile material ;**

When a ductile material (mild steel) bar of uniform cross-sectional area is subjected to a uniformly increasing load in a "Universal Testing Machine" till failure & when plotted, the stress-strain curve will be obtained as in fig.

A = Proportionality limit
B = Elastic limit
C = Upper yield limit
C' = Lower yield limit
D = Ultimate stress point
E = Breaking point

OA = Linear zone
OB = Elastic zone
BC = Plastic zone
CC' = Yielding zone
DE = Necking zone.

In the initial stage (portion ***OA***) is a straight line, where stress is directly proportional to strain & the material obeys Hooke's law. Point '***A'*** is called proportional limit.

If the stress is increased beyond this limit up to point 'B', then in portion '***AB***' Hooke's law is not obeyed, though the material be still elastic. Point ***'B'*** is called elastic limit.

Beyond the elastic limit (the portion ***BC***) the rate of increase of strain will be more even without further increase in load & strain is not fully recoverable when the load is removed i.e. material undergoes plastic deformation. The point ***'C'*** is called upper yield point & the stress is known as yield stress.

In the portion ***CC'*** the yielding commences & there is a drop of the load (stress) at the point C', immediately after yielding commences at C. The point ***'C'*** is called lower yielding point.

After yielding any further increase in load, will cause considerable increase in strain & the stress-strain curve continues to rise up to the point 'D'. The point 'D' is called ultimate stress point. At this point the stress is called ultimate stress. At this point '***D***' a local neck is formed.

In the portion ***DE***, the load is maximum until the fracture at 'E' takes place. The stress will decrease at 'E' & is named as breaking stress. The point '***E***' is called breaking point.

- **GENERALISED HOOKE'S LAW FOR TWO DIMENSIONAL or BI-AXIAL STRESS SYSTEM :**
  Considering an infinitesimal element at a point in a stressed body. It is subjected to tension in X & Y directions as in figure.
  Strain in X-direction
  = Strain due to σ_X in X-direction + strain due to σ_Y acting in X-direction.
  Or $\varepsilon_X = \frac{1}{E}(\sigma_X - \mu.\,\sigma_Y)$ . . . (1)
  Strain in Y-direction
  = Strain due to σ_Y in Y-direction + strain due to σ_X acting in Y-direction
  $\varepsilon_Y = \frac{1}{E}(\sigma_Y - \mu.\,\sigma_X)$ . . . (2)
  Strain in Z-direction
  = Strain due to σ_X in Z-direction + strain due to σ_Y acting in Z-direction
  $\varepsilon_Y = \frac{-\mu}{E}(\sigma_X + \sigma_Y)$ . . . (3)
  The above equations are known as generalized Hooke's Law for two dimensional stress system.
- **Volumetric strain of a rectangular body subjected to an axial force :**
  Vol. strain $\varepsilon_v = \delta V / V = \frac{P}{btE}(1- \frac{2}{m})$
- **Volumetric strain of a rectangular body subjected to three mutually perpendicular forces :**
  Vol. strain $\varepsilon_v = \delta V/V = \boldsymbol{\varepsilon_x + \varepsilon_y + \varepsilon_z}$
  Where, $\varepsilon_x = \frac{\sigma x}{E} - \frac{\mu.\sigma y}{E} - \frac{\mu.\sigma z}{E}$, $\varepsilon_y = \frac{\sigma y}{E} - \frac{\mu.\sigma z}{E} - \frac{\mu.\sigma x}{E}$
  & $\varepsilon_z = \frac{\sigma z}{E} - \frac{\mu.\sigma x}{E} - \frac{\mu.\sigma y}{E}$
- **ULTIMATE STRESS :**

The maximum axial load that can be applied on a member without causing the rupture or failure is called the ultimate load.

Ultimate stress is defined as the ratio of ultimate load to the initial cross-sectional area.

The maximum stress obtained in the stress-strain curve is called ultimate stress.

Mathematically, $\boldsymbol{\sigma_U = P_U/A}$

- **WORKING STRESS (ALLOWABLE STRESS) :**
  In designing a member, actual stress developed must be less than working stress.

When designing machine parts, it is desirable to keep the stress lower than the maximum or ultimate stress at which failure of material takes place. This stress is known as working stress or design stress.

Working stress $= \frac{maximum\ stress}{factor\ of\ safety}$ ------- In general

$= \frac{Yield\ stress}{Factor\ of\ safety}$ - - - - - For Ductile material

$= \frac{Ultimate\ stress}{Factor\ of\ safety}$ - - - - - For brittle & ductile material

❖ **PERCENTAGE OF ELONGATION :**

It is the ratio of increase in length to original length. It is expressed in percentage.

Percentage of elongation $= \frac{Increase\ \ in\ length}{Original\ \ length}$ X 100 %

❖ **PERCENTAGE REDUCTION IN AREA :**

It is the ratio of decrease in area to original area. It is expressed in percentage.

Percentage reduction in area $= \frac{Decrease\ \ in\ area}{Original\ \ area}$ X 100 %

- **FACTOR OF SAFETY :**
  It is the ratio of ultimate stress to working stress of a material.
  F.O.S $= \frac{Ultimate\ stress}{Working\ stress}$
- **0.2 % PROOF STRESS :**
  It is defined as the amount of stress required to increase 0.2 % of elongation or strain of the material.
  It is find out by drawing a parallel line with the proportional limit line, such that its gap is 0.2 % of strain axis. This line intersects the curve at certain point that value of stress is shown as 0.2 % proof stress.

- **STRAIN HARDENING :**
  If a material can be stressed considerably beyond the yield point without failure, it is said to be strain harden. This is true of many structural metals.
- ***PRINCIPLE OF SUPERPOSITION :***

It states that if a body is acted upon by a no. of loads on various segments of a body, then the net effect on body is the sum of the effects caused by each of the loads independently on the respective segment of the body.

Each segment can be considered for its equilibrium. This is done by making a diagram of the segment along with various forces acting on it. This diagram is referred as free body diagram.

While using this principle for an elastic body which is subjected to a number of direct forces (tensile or compressive) at different sections along the length of the body, first the free body diagram of individual section is drawn. Then the deformation of the each section is obtained. The total deformation of the body will be then equal to the algebraic sum of deformation of the individual sections.

$\delta l = \delta l_1 + \delta l_2 + \delta l_3 + \ldots\ldots$

$= (P_1 l_1)/(A_1 E) + (P_2 l_2)/(A_2 E) + (P_3 l_3)/(A_3 E) + \ldots\ldots$

It is applicable to the parameters like stress, strain & deformation, but it is not applicable to materials which don't follow Hooke's law.

- ***STATICALLY INDETERMINATE SYSTEM :***

When a system comprises 2 or more members of different materials, the forces in various members can't be determined by the principle of statics alone. Such systems are known as "statically indeterminate system".

(in such systems, additional equations are required to supplement the equation of statics to determine unknown forces.

- Usually these equations are obtained from deformation conditions of the system & are known as compatibility equation.
- **STRESSES IN MEMBERS WITH VARYING SECTION :**
  Consider a non-uniform cross-section of a member AB,BC & CD having cross-sectional areas A_1 , A_2 & A_3 with lengths L_1 , L_2 & L_3 as in fig.
  Even though the total force on each section is same, the intensities of stress will be different for three sections.
  For instance, Intensity of stress for the portion AB, $\sigma_1 = P / A_1$
  Intensity of stress for the portion BC, $\sigma_2 = P / A_2$
  Intensity of stress for the portion CD, $\sigma_3 = P / A_3$
  Let, E be the young's modulus
  Strain in the part AB, $\varepsilon_1 = \sigma_1/E$ Change in part AB, $\delta l_1 = \varepsilon_1 \cdot l_1$
  Strain in the part BC, $\varepsilon_2 = \sigma_2/E$ Change in part BC, $\delta l_2 = \varepsilon_2 \cdot l_2$
  Strain in the part CD, $\varepsilon_3 = \sigma_3/E$ Change in part AB, $\delta l_3 = \varepsilon_3 \cdot l_3$
  Total change in length of bar, $\delta l = \delta l_1 + \delta l_2 + \delta l_3 = (P\,l_1)/(A_1 E) + (P\,l_2)/(A_2 E) + (P\,l_3)/(A_3 E)$
- ***COMPOSITE BAR :***

A bar consisting of 2 or more bars of different materials in parallel is known as a composite or compound bar. In such a bar, the sharing of load by each can be found by applying equilibrium & the compatibility equation.

Consider a solid bar is enclosed in a hollow tube.

Let the subscripts 1 & 2 denote the solid bar & tube respectively.

Equilibrium Equation : as the total load must be equal to the load taken by individual members. $P = P_1 + P_2$

Compatibility Equation : the deformation of bar must be equal to the tube. i.e. $\delta l_1 = \delta l_2$

The strain in the solid tube = The strain in the tube. i.e. $\varepsilon_1 = \varepsilon_2$ (as length of bar = length of tube)

Or $\sigma_1/E_1 = \sigma_2/E_2$ or $\boldsymbol{\sigma_1 = \sigma_2 \cdot E_1/E_2}$ . . . which gives a relation between stresses of bar & tube.

Where E_1/E_2 = Modular ratio of 1^{st} material to 2^{nd} material.

From equilibrium equation, $P = P_1 + P_2 = \sigma_1.A_1 + \sigma_2.A_2$

By putting the value of σ_1 in terms of σ_2 in the above equation we can determine the value of σ_1 & σ_2.

Where, P = Total load carried by the composite bar.

P_1 = Load shared by the rod, And P_2 = Load shared by the tube,

σ_1 = Induced stress in the rod, σ_2 = Induced stress in the tube,

E_1 = Young's modulus of the rod, E_2 = Young's modulus of the tube,

A_1 = Cross-sectional area of the rod, and A_2 = Cross-sectional area of the tube.

- In composite sections, the stresses induced in the materials can be determined in the similar way as in the composite bar
- **STRESSES IN COMPOSITE SECTION :**

A composite member is composed of 2 or more different materials joined together in such a way that, the system is elongated or compressed as a whole unit.

In such a case, following two governing principles to be followed –

1. Elongation or contraction of individual materials of a composite member are equal. Hence the strains are induced in those materials are also equal. (as lengths are equal)

   $\boldsymbol{\varepsilon_1 = \varepsilon_2}$ . . . . compatibility equation. (From this equation we can get a relation between induced stresses of two materials.)
2. The sum of loads carried by individual members of composite is equal to the total load applied on composite.

   $\mathbf{P_c = P_1 + P_2}$ . . . . Equilibrium equation. (From this equation we can determine the values of the induced stresses)

- **MODULAR RATIO :** The ratio of E_1/E_2 is called modular ratio of the first material to the second. It is denoted by 'm'. ($\mathbf{m = E_1/E_2}$)

  for example, *Modular ratio of steel to concrete* = $\boldsymbol{E_S/E_C}$
- **EQUIVALENT AREA OF A COMPOSITE /COMPOUND SECTION :**

  In a concrete reinforced column,

  Equivalent concrete area, $\mathbf{A_c = A + (m - 1) A_s}$

  Where, A = total area of concrete column

  A_s = Area of steel & m = modular ratio

  Stress in concrete, $\boldsymbol{\sigma_c} = \dfrac{\textit{Load on column}}{\textit{Equivalent concrete area}}$

  Stress in steel, $\boldsymbol{\sigma_s = m.\sigma_c}$
- ***TEMPERATURE STRESSES :***

The length of a material which undergoes a change in temperature also changes & if the material is free to do so, no stresses are developed in the material.

However, if the material is constrained, stresses developed in the material which are known as ***Temperature stresses***.

Consider a bar of length L, if its temp. is increased through t°c, its length is increased by an amount **Lαt** i.e. δl = Lαt. Where, α = coefficient of thermal expansion

But if the bar is constrained & is prevented from expansion, the temperature stress is induced in the material which is generally,

Temperature stress, $\boldsymbol{\sigma = \alpha.t.E}$

Temperature strain, $\varepsilon = \delta l/l = l\alpha t/l = \alpha t$

As E = $\dfrac{temp.\ stress}{temp.\ strain} = \dfrac{\sigma}{\delta l/l} = \dfrac{\sigma}{l\alpha t/l} = \dfrac{\sigma}{\alpha t}$

Temp. stress = αtE

- ***COMPOUND SECTIONS :***

Consider a copper rod enclosed in a steel tube rigidly joined at each end.

Now, if the temp. is increased by 1^oc, the copper rod would tend to expand more as compared to steel tube. As the two are joined together, the copper is prevented from full expansion & is put in compression. The final position of the compound bar will be as in fig.

Let, σ_s = tensile stress in steel A_s = cross-sectional area of steel tube

σ_c = compressive stress in copper A_c = cross sectional area of copper rod.

From **equilibrium equation**, Tensile force in steel tube = compressive force in copper rod

i.e. $\mathbf{\sigma_s. A_s = \sigma_c. A_c}$ **or** $\mathbf{\varepsilon_s . E_s . A_s = \varepsilon_c . E_c . A_c}$ **as** $\mathbf{\sigma = \varepsilon . E}$

Compatibility Equation :

Let α_s = co-efficient of thermal expansion in steel

α_c = co-efficient of thermal expansion in copper.

Now elongation of steel tube (due to temp. + due to tensile stress)

= Elongation of copper rod (due to temp. – due to comp. stress)

or Temp. strain of steel + tensile strain = Temp. strain of copper – comp. strain

i.e. $\boldsymbol{\alpha_s \cdot t \cdot E_s + \sigma_s = \alpha_c \cdot t \cdot E_c - \sigma_c}$

$\boldsymbol{\alpha_s \cdot t + \sigma_s/E_s = \alpha_c \cdot t - \sigma_c/E_c}$

$\boldsymbol{\alpha_s \cdot t + \varepsilon_s = \alpha_c \cdot t - \varepsilon_c}$

$\boldsymbol{\varepsilon_s + \varepsilon_c = \alpha_c \cdot t - \alpha_s \cdot t}$

$\boldsymbol{\varepsilon_s + \varepsilon_c = (\alpha_c - \alpha_s)\, t}$

- ***SHRINKING ON :***

  A thin tyre of steel or any other metal can be shrunk in to wheels of slightly larger dia. by heating the tyre to a certain degree which increases its diameter. When the tyre has been mounted & the temp. falls to the normal temp., the steel tyre tends to come to its original diameter & thus tensile (hoop0 stress is setup in the tangential direction.

Let d & D be the diameters of the steel tyre and of the wheel on which the steel tyre is to be mounted,

then the strain, **ε** = $\frac{\pi D - \pi d}{\pi d} = \frac{D-d}{d}$

Circumferential tensile stress or ***hoop stress = ε . E*** = $\left(\frac{D-d}{d}\right) . E$

- ***COMPLIMENTARY SHEAR STRESS :***

Consider an infinitely small rectangular element ABCD under shear stress of intensity 'τ' acting on planes AD & BC.It is clear from fig. that, the shear stress acting on element will tend to rotate the block in clockwise direction. As there is no other force acting on the element, static equilibrium of element can only be obtained, if another couple of same magnitude is applied in counter clockwise direction.

This can be achieved by having shear stress of intensity ' τ ' on the faces AB & CD.

Let length of AB = x & length of BC = y & unit thickness.

The force of given couple = **τ (y . 1) = τ . y**

And moment of given couple = **(τ . y) x**

Similarly force of balancing couple **= τ'(x . 1) = τ'x**

And moment of balancing couple = **τ' x y**

For equilibrium, equating the two, **τ y x = τ' x y or τ = τ'**

It shows that, magnitude of balancing shear stress is the same as of the applied stress. The shear stress on the transverse pair of faces are known as complimentary shear stresses. Thus every shear stress is always accompanied by an equal complimentary shear stress on perpendicular planes.

- ***STRAIN ENERGY :***

Strain energy may be defined as the energy, which is developed in a elastic material due to straining by an external force (i.e. due to work done on it to produce an elastic deformation)

Strain energy stored, U = work done by external force.

- Strain energy stored in an elastic body not only due to axial stress, but also due to shear stress, bending stresses.

- ***RESILIENCE :*** The total strain energy stored in a strained member is termed as 'resilience'.
- ***PROOF RESILIENCE :*** It is defined as the maximum strain energy that can be stored in an elastic member without exceeding its elastic limit i.e. without causing any permanent deformation.
- ***PROOF STRESS :*** The stress induced in an elastic body when it possesses maximum strain energy is termed as its proof stress.
- ***MODULUS OF RESILIENCE :*** It is defined as the greatest amount of strain energy per unit volume that an elastic material can absorb without exceeding its elastic limit.

In other words, for any elastic member, its proof resilience per unit volume is termed as 'Modulus of Resilience' of the material of the member.

Modulus of resilience = $\frac{\textit{Proof Resilience}}{\textit{Volume of the body}}$

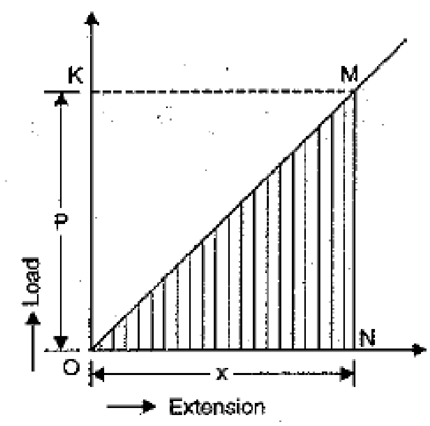
- ***TYPES OF LOADING :***
- **Gradual Loading –**In this case, loading starts from zero & increases linearly till the body is fully loaded.
- **Suddenly applied loading –**In this case, the applied force or load is suddenly applied on the body such that the force or load remains constant throughout the process of deformation of the body.
- **Impact Loading –**In this case, the external force or load is allowed to fall freely on a body from a certain height with an impact.
- ***STRAIN ENERGY IN TENSION & COMPRESSION FOR VARIOUS TYPES OF LOADING :***

- ***Gradually Applied Loads :***

In this type of loading, the external force applied on the body starts from zero & increases linearly until the body is fully loaded.

The average force is half the force actually applied on the body.

Let us consider an elastic bar, subjected to a gradually applied load.



Let, P = Gradually applied load at elastic limit.
A = Cross-sectional area of bar
L = Length of the bar
f or σ = axial stress induced on the bar
e = axial strain produced in the bar
δl = deformation of the bar up to elastic limit
E = Modulus of elasticity of material of the bar.

Then average load applied to the material $= \frac{0+P}{2} = \frac{P}{2}$

Work done by external force up to elastic limit = Area of the shaded portion in the figure

= Average force x deformation $= \frac{P}{2} . \delta l$ = proof resilience

But, $P = \sigma . A$ & $\delta l = \frac{P.L}{A.E} = \frac{\sigma L}{E}$

External work done $= \frac{\sigma.A}{2} \times \frac{\sigma.L}{E} = \sigma^2 . AL/2E = \sigma^2 . V/2E$ where, V = volume of bar = A . L

Strain energy stored in the bar, U = External work done = $\sigma^2 . V/2E$ = Proof resilience

Modulus of resilience of material of bar $= \frac{Proof\ Resilience}{Volume} = (\sigma^2 . V)/(2E.V) = \sigma^2/2E$

- ***Suddenly Applied Load :***

In this type of loading, the applied force remains constant throughout the process of deformation. Hence, the average force is equal to the actual force applied on the body.

Let us consider an elastic bar subjected to a suddenly applied load 'P'.

Since for suddenly applied load, the applied force remains constant throughout the process of deformation, the plot will be parallel to x-axis.

Now external work done = Area of shaded portion = **P . δl**

But we know that, the strain energy stored, $U = (\sigma^2 . AL)/2E$

Since, external work done = Strain energy stored

$$P.\delta l = (\sigma^2 . AL)/2E \quad \text{or} \quad P . \left(\frac{\sigma L}{E}\right) = (\sigma^2 . AL)/2E \quad \text{or} \quad \sigma = 2\frac{P}{A}$$

If 'P' is applied gradually, stress setup $= \frac{P}{A}$

Hence, the instantaneous stress induced in a member ***due to***

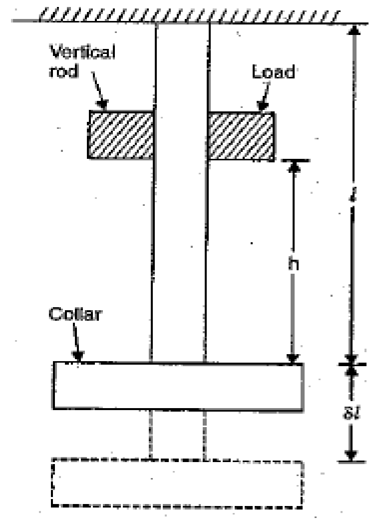
***(a suddenly applied load is twice that when the load is gradually applied*.)**

- ***Impact Loading :***

  When a weight (external force) is dropped on a member from some height, impact loading takes place.

  In fig. an elastic bar is to be suspended from the ceiling.

A collar is fixed at its bottom end & weight ‘P’ is allowed to slide freely along the bar.



Let, P = The sliding weight

L = Length of the bar

A = Cross-sectional area of the bar

f or σ = Axial stress induced in the bar due to falling of the weight ‘P’ on the collar with impact.

δl = Maximum instantaneous elongation of bar as a result of impact loading.

h = height of free fall of weight (P) above collar.

We know that,

work done by external force = Force x distance

$= \mathbf{P\ .\ (h\ +\ \delta l\ )}$

Strain energy stored in the bar = $\mathbf{(\sigma^2 A L)/2E}$

Since, strain energy stored = work done by external force

We have, $\mathbf{(\sigma^2 A L)/2E = P(h + \delta l)}$

$$(\sigma^2 A L)/2E = P\left(h + \frac{\sigma L}{E}\right)$$

PRINCIPAL STRESSES AND STRAINS

PLANE STRESS / 2-DIMENSIONALSTRESS CONDITION:

If a cubical element considered within a strained element is under the action of stresses acting on only two pairs of parallel planes & the third pair of parallel planes is free from any stress, the element is said to be under plane stress (2-dimensional stress) condition.

Let us consider a cubical element ABCD within a strained material subjected to normal stresses σ_1 & σ_2 & shear stress τ.

The first pair of parallel planes AD & BC is under normal tensile stress σ_1 & shear stress τ.

The second pair of parallel planes AB & DC, is under normal stress σ_2 & shear stress τ.

But, on the third pair of parallel planes ABCD, there is no stress. Hence, the cubical element is under plane stress condition.

PRINCIPAL PLANES :

When a material is under plane stress condition, it is observed that through any arbitrary point in the material, there exist two mutually perpendicular planes in which the normal stresses are optimum (i.e. maximum & minimum). These planes are termed as principal planes. It can be proved that no shear stress acts on these planes.

PRINCIPAL STRESSES :

The optimum normal stresses acting on the principal planes are termed as principal stresses.

Among the two principal stresses, one is maximum & the other is minimum. The principal stress having the maximum value is called the major principal stress & that having the minimum value is called the minor principal stress.

Similarly, the planes carrying the major & minor principal stresses are called major & minor principal planes respectively.

METHODS FOR DETERMINING STRESSES ON OBLIQUE SECTIONS :

There are two methods for finding stresses on oblique sections of a strained body –

- Analytical method, and
- Graphical method.
  - ❖ (*Oblique plane is a plane which is inclined to the length of the object at certain angle other than at right angle*)

SIGN CONVENTIONS FOR ANALYTICAL METHOD :

- All the tensile stresses & strains are taken as positive, where as all the compressive stresses & strains are taken as negative.
- The shear stress which tends to rotate the element in the clockwise direction is taken as positive, where as that which tends to rotate in an anti-clockwise direction as negative.

➢ ***STRESSES ON AN OBLIQUE SECTION OF A BODY SUBJECTED TO A DIRECT STRESS IN ONE PLANE :***

➢ ***STRESSES ON AN OBLIQUE SECTION OF A BODY SUBJECTED TO DIRECT STRESSES IN TWO MUTUALLY PERPENDICULAR DIRECTION :***

- ***GRAPHICAL METHOD FOR THE STRESSES ON AN OBLIQUE SECTION OF A BODY :***
  SIGN CONVENTIONS :

1. The angle is taken with reference to x-x axis. All the angles traced in the anti-clockwise direction to x-x axis, are taken as negative. Where as those in the clockwise direction as positive.
The value of θ, until & unless mentioned is taken as positive & drawn clockwise (fig. – 1)

2. The measurements above x-x axis & to the right of y-y axis are taken as positive, where as those below x-x axis & to the left of y-y axis as negative.

3. Sometimes there is a little variation in the results obtained by analytical method & graphical method.
The values obtained by graphical methods are taken to be correct, if they agree up to the first decimal point with values obtained by analytical method.
e.g in analytical – 8.66 & 4.32
in graphical – 8.7 & 4.3

➢ ***MOHR'S CIRCLE FOR STRESSES ON AN OBLIQUE SECTION OF A BODY SUBJECTED TO A DIRECT STRESS IN ONE PLANE :***

Consider a rectangular body of uniform cross-sectional area & unit thickness subjected to a direct tensile stress along x-x axis.

Now let us consider an oblique section AB inclined with x-x axis on which we are required to find out the stresses.

Let, σ = tensile stress in x-x direction.
θ = angle which the oblique section AB makes with the x-x axis.

Consider the equilibrium of the wedge ABC. Now draw the Mohr's circle of stresses as discussed below :

1. Take some suitable point O & through it draw a horizontal line XOX.
2. Cut-off OJ equal to the tensile stress(σ) to some suitable scale & towards right (as σ is tensile)
3. Bisect OJ at C. now the point O represents stress system on plane BC & the point J represents the stress system on plane AC.
4. Now with C as centre & radius equal to CO or CJ, draw a circle. It is known as Mohr's circle for stresses.
5. Now through C draw a line CP making an angle of 2θ with CO in clockwise direction meeting the circle at P. The point P represents the section AB.
6. Through P draw PQ perpendicular to OX. Join OP.
7. Now OQ = Normal stress

QP = Shear stress

& OP = Resultant stress to the scale & the angle POJ is called the angle of obliquity (θ)

PROOF :

- ***MOHR'S CIRCLE FOR STRESSES ON AN OBLIQUE SECTION OF A BODY SUBJECTED TO DIRECT STRESSES IN TWO MUTUALLY PERPENDICULAR DIRECTIONS :***

Consider a rectangular body of uniform cross-sectional area & unit thickness subjected to direct tensile stresses in two mutually perpendicular directions along x-x & y-y axes.

Now let us consider an oblique section AB inclined with x-x axis on which we are required to find out the stresses.

Let, σ_x = Tensile stress in x-x direction (major tensile stress)

σ_y = Tensile stress in y-y direction (minor tensile stress)

θ = Angle which the oblique section AB makes with x-x axis in clockwise direction.

Consider the equilibrium of wedge ABC. Now draw the Mohr's circle of stresses as discussed below :

1. Take some suitable point O & draw a horizontal line XOX.
2. Cut off OJ & OK equal to the tensile stresses σ_x & σ_y to some suitable scale towards right (as both stresses are tensile). The point J represents the stress system on plane AC & the point K represents the stress system on plane BC.
3. Bisect JK at C.
4. Now C as centre & radius equal to CJ or CK draw a circle. It is known as Mohr's circle of stresses.
5. Now through C, draw a line CP making an angle 2θ with CK in clockwise direction meeting the circle at P. The point P represents the stress system on section AB.
6. Through P draw perpendicular PQ to line OX. Join OP.
7. Now OQ = Normal stress,

   QP = Shear stress

& OP = Resultant stress to the scale.

CM CN gives the maximum shear stress to the scale. The angle POC is called angle of obliquity.

PROOF :

SHEAR FORCE & BENDING MOMENT

- ***BEAM :*** A beam is a horizontal structural member subjected generally to vertical loads.

(Any member of a machine or structure whose one dimension (length) is very large as compared to the other two dimensions (width & thickness) & which can carry lateral or transverse loads in the axial plane is called a beam).

The amount & extent of external load which a beam can carry depends upon –

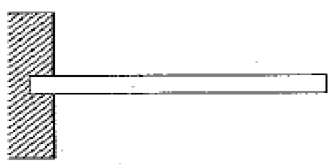
- The distance between supports & the overhanging lengths from supports.
- The type & intensity of loading.
- The type of supports, and
- The cross-section & elasticity of beam.

- ***CLASSIFICATION OF BEAMS :*** Beams are classified as –

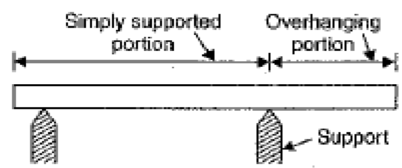
- **Simply Supported Beams :**
  A beam which rests on two supports placed at its ends is known as 'simply supported beam'.
- **Cantilever Beam :**
  A beam which is fixed at one end & free at the other end is known as 'cantilever beam'
- **Overhanging Beam :**
  If some length of a beam extends beyond its supports, it is said to be an 'overhanging beam'. This beam may have overhanging both sides or on one side only.
- **Fixed Beam or Built-in Beam :**
  If a beam is fixed at both ends, it is called a 'fixed beam'.
- **Propped Cantilever Beam :**
  When a support is provided at some suitable point of a cantilever beam in order to resist the deflection of beam, it is known as 'propped cantilever beam'.
- **Continuous Beam :**
  If a beam is supported on more than two supports, it is said to be a 'continuous beam'.

❖ The three types of beam i.e. simply supported, cantilever & overhanging beams are known as statically determinate beam, because support reactions of these beams are determined by use of equation of static equilibrium.

(S.S.Beam)

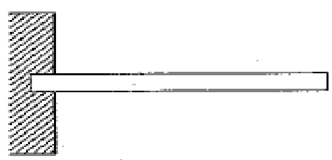


(Cantilever Beam)

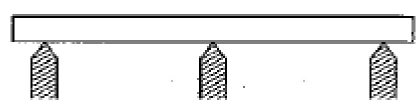


(Overhanging Beam)

(Fixed Beam)



(Propped Cantilever Beam)



(Continuous Beam)

❖ The other 3 types of beam i.e. fixed, propped cantilever & continuous beams are known as statically indeterminate beam, because support reactions of these beams can't be determined by use of equation of static equilibrium.

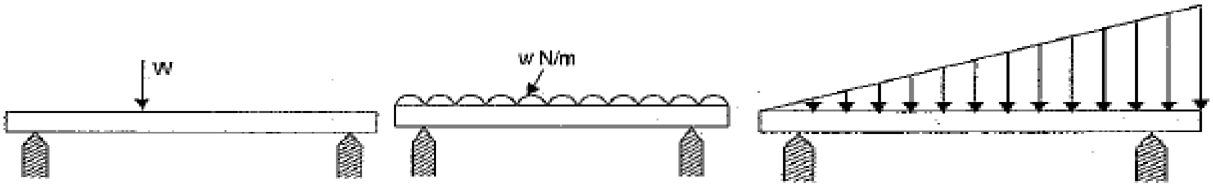
❖ Equations of equilibrium are - $\Sigma F_x = 0, \ \Sigma F_y = 0 \ \& \ \Sigma M = 0$

- ***TYPES OF LOAD ON A BEAM :*** The following 3 types of loading may act on a beam –
- **Concentrated load / point load /Isolated load :**

The load which acts at a point on the beam is known as 'concentrated load' & are denoted by N or kN.

- **Uniformly Distributed Load :**

An uniformly distributed load is one which is spread uniformly over a beam so that each unit of length is loaded with same amount of load & denoted by N/m or kN/m. A common example of UDL is self-weight of the beam.



(POINT LOAD) *(U.D.LOAD)* *(U.V. LOAD)*

- **Uniformly Varying Load / Gradually Varying Load :**

  The load which is spread over a beam, in such a way that its intensity (i.e. load/unit length) varies uniformly from a minimum value at one end to a maximum value at the other end.
- ❖ If the load varies uniformly from zero at one end to a maximum value at the other end, the load is known as 'Triangular load'.
- ❖ In case of UDL & UVL, the area of load diagram gives the total value of the load.
- ***SHEAR FORCE :***

Shear force is the unbalanced vertical force on one side (to the left or right) of a beam & is the sum of all the normal forces on one side of the section.

It also represents the tendency of either portion of beam to slide or shear laterally relative to the other.

- A force at a section means, a force of a certain magnitude acting at that point whereas, a shear force at a section means, the sum of all the forces on one side of the section.

  Consider the beam as shown in fig(a), it is simply supported at two ends & carries three loads. The reactions at the supports are R_1 & R_2.

Now if the beam is imagined to cut at section x-x into two portions (as in fig(b)), the resultant of all the forces (loads as well as reaction of support) to the left of the section is F (assuming upwards).

i.e. $\mathbf{F_1 = R_1 - W_1 - W_2}$ . . . . (1) net vertical load on left side.

As the beam is in equilibrium, the resultant of the forces to the right of x-x must also be F (downwards)

i.e. $\mathbf{F_2 = R_2 - W_3}$

again for equilibrium, $\mathbf{R_1 + R_2 = W_1 + W_2 + W_3}$ **i.e.** $\mathbf{R_1 - W_1 - W_2 = W_3 - R_2 = -(R_2 - W_3)}$

i.e. $\mathbf{F_1 = -F_2}$

the forces F_1 & F_2 are equal & opposite & act across the section x-x axis tending to shear it off & are known as shear forces.

SIGN CONVENTIONS :

- ***Shear force is positive*** if the net resultant force to the left of the section ***is upward*** & Shear force is ***negative,*** if the net resultant force to the left of the section is ***downward.***

- ***BENDING MOMENT :***

The bending moment at any point along a loaded beam is the algebraic sum of the moments of all the vertical forces acting to one side of the point about the point.

Refer to the figure, the clockwise moment at this section due to all loads acting on the beam to the left of the section. $M_X = R_A . x - W_1(x - b) - W_2(x - b)$ . . . . . (1)

If we consider the forces to the right of the section X-X, then anticlockwise moment is

$M_X = -R_B(l - x) + W_3(l - x - c)$ . . . . . (2)

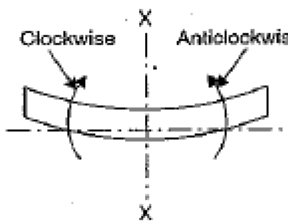
For equilibrium of beam, the equations (1) & (2) are equal.

Bending moment at any section of a beam is the algebraic sum of the moments about the section of all the forces (including reaction) acting on the beam either to the left or to the right of the section.

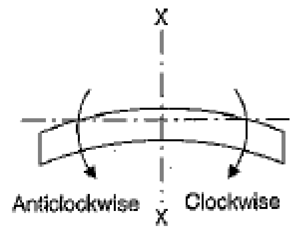
- ***Sign Conventions :***

A Bending Moment is considered positive, if it produces compression on top fibers of the beam & negative, if it produces tension on the top fibers of the beam.

Positive B.M is called "sagging" & negative B.M is called "hogging".

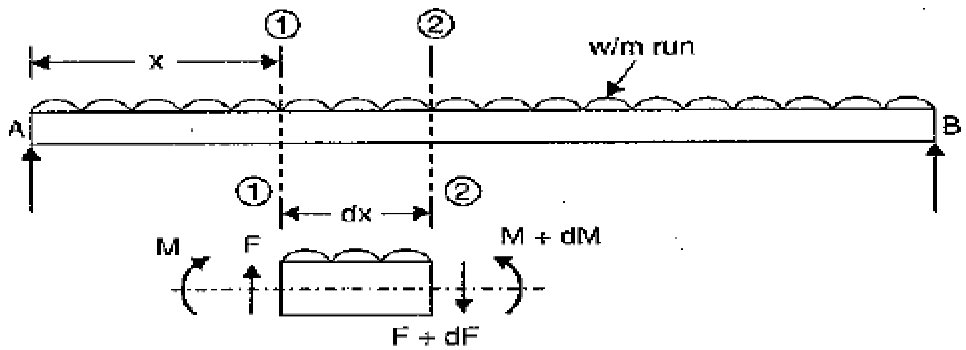


(SAGGING) – POSITIVE B.M.



(HOGGING) – NEGATIVE B.M.

RELATION BETWEEN LOAD, SHEAR FORCE & BENDING MOMENT :



Consider a beam carrying an UDL ω per unit length.

Now consider a small length δx of the beam between two sections A-A' & B-B' as in fig. carrying an UDL ω per unit length.

Let the shear force & bending moment at the section A-A' are F & M respectively.

Similarly, the S.F & B.M at the section B-B' are (F + δF) & (M + δM) respectively.

Since section in between A-A' & B-B' is in equilibrium, the sum of all vertical forces on it must be in equilibrium.

$F + W.\delta x = F + \delta F$ i.e. $\delta F = W.\delta x$ or $\frac{\delta F}{\delta x} = W$

The rate of change of shear force is equal to the intensity of loading.

Again equating moments of all external forces on δx, about the centre of δx.

$$M + F.\frac{\delta x}{2} + (F + \delta F)\frac{\delta x}{2} = M + \delta M$$

$$M + F.\frac{\delta x}{2} + F.\frac{\delta x}{2} + \frac{\delta F.\delta x}{2} = M + \delta M$$

$$M + F.\delta x + \frac{\delta F.\delta x}{2} = M + \delta M$$

Neglecting small quantities we get, $M + F.\delta x = M + \delta M$ *or* $\frac{\delta M}{\delta x} = F$

Hence, the rate of change of bending moment along an uniformly loaded beam is equal to shear force.

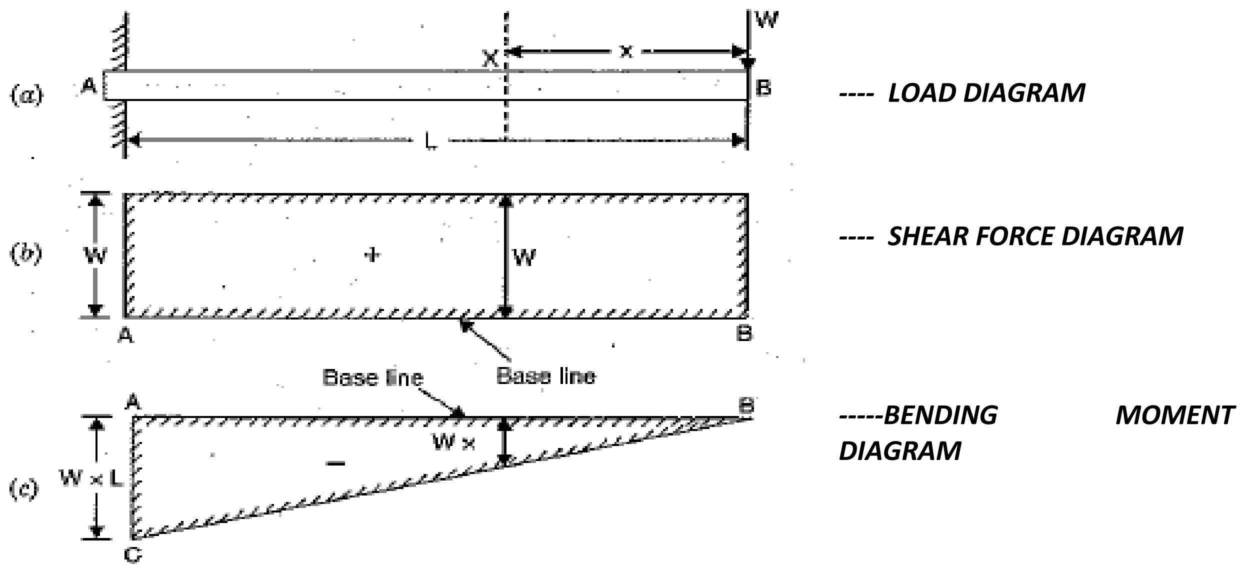
For maximum value of M, $\frac{\delta M}{\delta x} = 0$ i.e. $F = 0$

So, B.M. is maximum, where S.F. is zero.

- ***IMPORTANT POINTS FOR DRAWING S.F & B.M DIAGRAM :***

1. The positive values of S.F & B.M are plotted above the base line & negative values below the base line.
2. For vertical loads & reactions vertically up or down lines are drawn in S.F.D.
3. The S.F between two vertical loads only remain constant & for that a horizontal line is drawn between two load points. But B.M varies at a constant rate according to the distance & for that an inclined straight line is drawn between the load points.
4. For UDL between two points S.F.D will be inclined straight line & B.M.D will be parabolic curve.
5. For a beam consisting of both concentrated load & UDL, the S.F.D will be inclined line for UDL & vertically up or down lines for concentrated load or reaction. For B.M.D shall be a combination of parabolic curve & inclined straight line i.e. shall be flatter & be drawn in between inclined straight line & parabolic curve.
6. For beam subjected to UVL, the S.F.D & B.M.D will be parabolic & cubic curve respectively.
7. The B.M. at the supports of a S.S.Beam is zero. Also the B.M. at free end of a cantilever is zero.
8. The B.M. is maximum at the section, where S.F. changes sign.
9. If not otherwise mentioned specially, self weight of beam is to be neglected.
10. In case of overhanging beam, the maximum B.M. is equal to the negative maximum B.M.

- ***CANTILEVER BEAM WITH CONCENTRATED LOAD AT FREE END :***
-



(The graph is inclined straight line due to 1st Degree equation)

Consider a cantilever beam of AB length L, carrying a point load W at its free end. Let us draw S.F.D & B.M.D after calculation.

- **SHEAR FORCE :**

S.F. at B, $\mathbf{F}_{B(\text{just right})} = 0$

$\mathbf{F}_{B(\text{just left})} = 0 + W = W$ (Positive sign due to right side downward)

S.F. at A, $F_A = W$

Maximum shear force occurs at the support = W

- **BENDING MOMENT :**

B.M. at B, $M_B = 0$

B.M. at a distance x, from B.

$M_X = -W.x$ (Negative sign due to right clockwise)

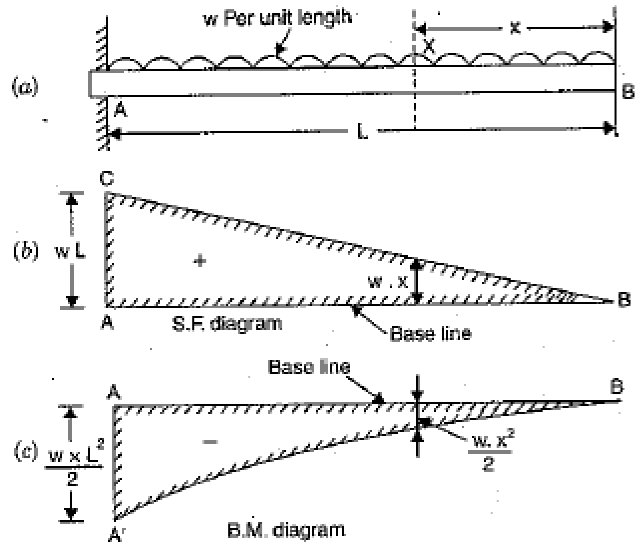
$M_A = -W.L$

Maximum B.M. occurs at support = W.L

- *For a cantilever beam, we have to consider the moments of all forces acting from the free end towards the fixed end for all cases, because the reaction at fixed end is unknown.*

- ***CANTILEVER WITH UDL :***

Figure shows a cantilever beam AB of length L carrying an UDL w/unit length over the entire length. Consider a section X-X at a distance x from free end.



- **SHEAR FORCE :**

  S.F. at B, $F_B = 0$

  S.F. at X-X, $F_X = w.x$

  (Positive sign due to right side downward)

  S.F. at A, $F_A = w.L$

Maximum shear force occurs at support A = w.L

- **BENDING MOMENT :**

  B.M. at B, $M_B = 0$

  B.M. at X-X, $M_X = -\frac{1}{2}w.x^2$

  B.M. at A, $M_A = -\frac{1}{2}w.L^2$

Maximum B.M. occurs at fixed end A = $-\frac{1}{2}w.L^2$

The graph is parabolic in shape due to 2^{nd} degree equation.

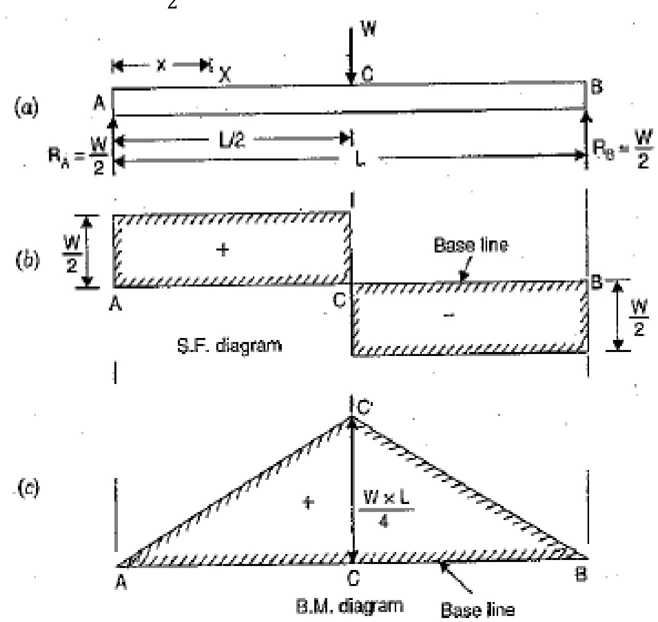
- ***SIMPLY SUPPORTED BEAM :***

PROCEDURE :

1. Calculate the reaction by taking moments of all forces about a support (generally left hand support). Then find reaction at other support (by total downward load – known reaction)
2. Calculate the S.F under all load points then draw S.F.D.(above or below the base line)
3. B.M of any reaction is the algebraic sum of moments of all forces about the section acting either on left hand side or right hand side of the section. The selection of the side should be made where the no. of forces is lesser. For checking, moments on either side of the section must be same.
4. Maximum B.M will occur where S.F.D changes sign from positive to negative.

- ***S.S.BEAM WITH A POINT LOAD AT ITS MID POINT :***

From symmetry of loading, the reactions are equal. i.e. $R_A = R_B = \frac{W}{2}$



- **SHEAR FORCE :**

  Portion AC -

  S.F at A, $\mathbf{F}_{A\ (\text{just left})} = 0$

  S.F at A, $\mathbf{F}_{A\ (\text{just right})} = R_A = \frac{W}{2}$

  S.F. at C, $\mathbf{F}_{C\ (\text{just left})} = \frac{W}{2}$

  Portion CB –

  S.F. at C, $\mathbf{F}_{C\ (\text{just right})} = \frac{W}{2} - W = -\frac{W}{2}$

  (in section BC, -ve sign due to left side is downward)

  S.F. at B, $\mathbf{F}_{B\ (\text{just left})} = -\frac{W}{2}$

  S.F. at B, $\mathbf{F}_{B\ (\text{just right})} = -\frac{W}{2} + R_B$

  $= -\frac{W}{2} + \frac{W}{2} = 0$

- **BENDING MOMENT :**

  $M_A = 0$

  $M_C = \frac{W}{2} . \frac{L}{2} = \frac{WL}{4}$ *(Positive sign due to left side clockwise)*

  $M_B = \frac{W}{2} .L - W.\frac{L}{2} = 0$

Maximum B.M. at C, $M_C = \frac{WL}{4}$

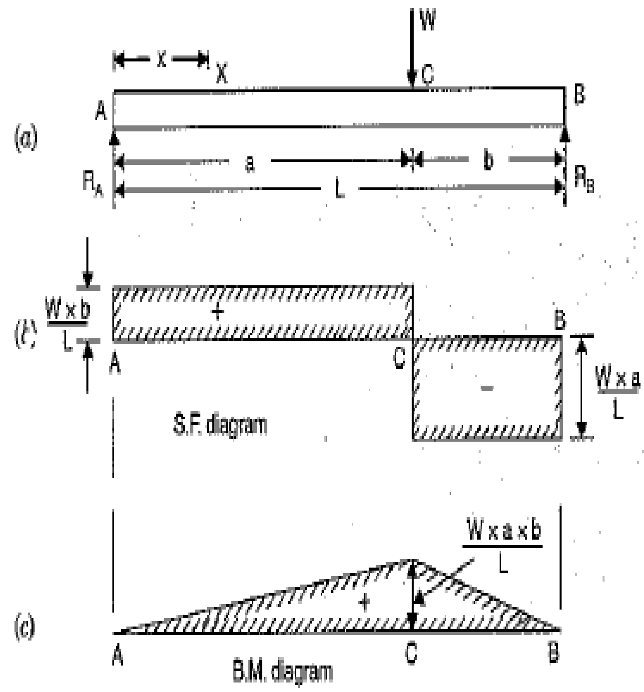
➢ *S.S. BEAM WITH A POINT LOAD OFF MIDSPAN OR WITH AN ECCENTRIC POINT LOAD :*

Consider a S.S.Beam AB of length L carrying a point load W at C.

$\Sigma F_Y = 0$ i.e. $R_A + R_B = W$

Taking moment about A, $R_B \cdot L = W \cdot a$ i.e. $R_B = \frac{W.a}{L}$

$R_A = W - R_B = W - \frac{W.a}{L} = \frac{W(L-a)}{L} = \frac{W.b}{L}$ as $a + b = L$



- **SHEAR FORCE :**

Shear force at A, $F_{A\,(\text{just left})} = 0$

$F_{A\,(\text{just right})} = 0 + R_A = \frac{W.b}{L}$

Shear force from A to C remains constant $= \frac{W.b}{L}$,

i.e. $F_{C\,(\text{just left})} = \frac{W.b}{L}$

$F_{C\,(\text{just right})} = \frac{W.b}{L} - W = \frac{W(b-L)}{L}$

$= \frac{-W(L-b)}{L} = \frac{-W a}{L}$

(-ve sign due to left side downward)

Shear force from C to B remains constant $= \frac{-W a}{L}$

$F_{B\,(\text{just right})} = \frac{-W a}{L} + R_B = \frac{-W a}{L} + \frac{W a}{L} = 0$

(+ve sign due to left side upward)

- **BENDING MOMENT :**

$M_A = 0$

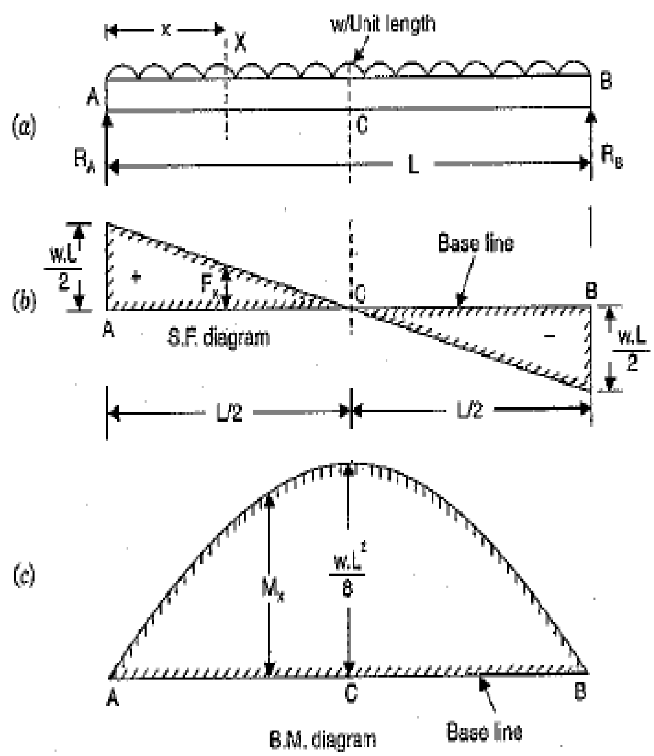
$M_C = \frac{W b}{L} \cdot a = \frac{W a.b}{L}$

$M_B = \frac{W b}{L} \cdot L - W.b = 0$

Maximum bending moment at C, as S.F. at C changes sign from +ve to –ve. $\mathbf{M_C = \frac{W\,a.b}{L}}$

➢ *S.S.BEAM WITH AN UDL :*

Consider a S.S.Beam AB of length L, carrying an UDL of w/unit length over the entire length. R_A & R_B are the reactions at the support A & B.



Total load on beam $= w.L$

$\Sigma F_Y = 0$ i.e. $R_A + R_B = w.L$

$\Sigma M = 0$, Taking moments about A, $R_B.L = w.L.\frac{L}{2}$

i.e. $R_B = \frac{wL}{2}$

$R_A = w.L - \frac{wL}{2} = \frac{wL}{2}$

- **SHEAR FORCE :**

Shear force at A, $F_{A\,(\text{just left})} = 0$

$F_{A\,(\text{just right})} = 0 + R_A = 0 + \frac{wL}{2} = \frac{wL}{2}$

(+ve sign due to left upward)

S.F at any section X-X at a distance of x from A.

$F_X = R_A$ – Total load upto X-X

$= R_A - w.x = \frac{wL}{2} - w.x = w\left(\frac{L}{2} - x\right)$

$F_{A\,(x=0)} = w\left(\frac{L}{2} - 0\right) = \frac{wL}{2}$

$F_{mid\,(x = L/2)} = w\left(\frac{L}{2} - \frac{L}{2}\right) = 0$

$F_{B\,(x = L)} = w\left(\frac{L}{2} - L\right) = -\frac{wL}{2}$

i.e. $F_{B\,(\text{just left})} = -\frac{wL}{2}$ and

$F_{B\,(\text{just right})} = -\frac{wL}{2} + \frac{wL}{2} = 0$

- **BENDING MOMENT :**

  B.M. at section X-X at a distance of x from A, $\mathbf{M_X} = R_A \cdot x - (w.x).\frac{x}{2} = \frac{w.L}{2} \cdot x - (w.x).\frac{x}{2} = \frac{w.x}{2}(L - x)$

  (Graph of B.M.D is parabolic due to 2nd degree equation)

  $\mathbf{M_A} = 0$

  $\mathbf{M_{MID} = \frac{w.L}{4}\left(L - \frac{L}{2}\right) = w\,L^2/8}$

  $\mathbf{M_B} = \frac{w.L}{2}.L - w.L.\frac{L}{2} = 0$

  Maximum B.M. will occur at mid point, where S.F. changes it's sign from **+ve to –ve**

- ***POINT OF CONTRAFLEXURE or POINT OF INFLEXION :***

Point of contraflexure occurs for only overhanging beams on one side or both the sides or when moment develops at support.

The point where the B.M<Diagram cuts the base after changing its sign from positive to negative or vice versa, is termed as the point of contra-flexure.

At this point, the bending of beam changes from sagging to hogging or vice-versa, therefore, B.M is zero.

❖ To locate point of contra-flexure, the expression for B.M at a distance of x from the end support is equalized to zero. Finding the value of x from there, the point of contra-flexure can be easily located.

- ***OVERHANGING BEAM :*** An overhanging beam is that which is projecting beyond the supports. Such like beams are encountered in residential buildings.

➢ ***OVERHANGING BEAM WITH A POINT LOAD :***

Consider a beam of length L having overhangs at CA & BD, each equal to 'a'. it carries concentrated loads W_1, W_2 & W_3 as in figure. R_A & R_B are the reactions at the support A & B.

$\mathbf{\Sigma F_Y = 0}$ i.e. $R_A + R_B = W_1 + W_2 + W_3$

$\mathbf{\Sigma M = 0}$, taking moment about B, $\mathbf{\Sigma M_B = 0}$

$R_A.L + W_3.a = W_1(L + a) + W_2\left(\frac{L}{2}\right)$

$R_A = W_1\left(\frac{L+a}{L}\right) + W_2.\frac{1}{2} - W_3\left(\frac{a}{L}\right)$

$R_B = (W_1 + W_2 + W_3) - R_A = W_1\left(1 - \frac{L+a}{L}\right) + W_2\left(1 - \frac{1}{2}\right) + W_3\left(1 - \frac{a}{L}\right) = W_1\left(\frac{-a}{L}\right) + W_2\left(\frac{1}{2}\right) + W_3\left(\frac{L-a}{L}\right)$

- **SHEAR FORCE :**

Shear force at C, $\mathbf{F_{C\,(just\ left)}} = 0$

$\mathbf{F_{C\,(just\ right)}} = 0 - W_1 = -W_1$

Shear force from C to A remains constant $= -W_1$

Shear force at A, $\mathbf{F_{A\,(just\ left)}} = -W_1$

$\mathbf{F_{A\,(just\ right)}} = -W_1 + R_A$

Shear force from A to E remains constant $= -W_1 + R_A$

Shear force at E, $\mathbf{F_{E\,(just\ left)}} = -W_1 + R_A$

$\mathbf{F_{E\,(just\ right)}} = -W_1 + R_A - W_2 = R_A - (W_1 + W_2)$

Shear force from E to B remains constant $= R_A - (W_1 + W_2)$

Shear force at B, $\mathbf{F_{B\,(just\ left)}} = R_A - (W_1 + W_2)$

$\mathbf{F_{B\,(just\ right)}} = R_A - (W_1 + W_2) + R_B$

$= (R_A + R_B) - (W_1 + W_2) = W_3$

Shear force from B to D remains constant $= W_3$

Shear force at D, $\mathbf{F_{D\,(just\ left)}} = W_3$

$\mathbf{F_{D\,(just\ right)}} = W_3 - W_3 = 0$

- **BENDING MOMENT :**

  $\mathbf{M_C} = 0$

  $\mathbf{M_A} = -W_1 . a$

  $\mathbf{M_E} = -W_1\left(a + \frac{l}{2}\right) + R_A\left(\frac{L}{2}\right)$

  $\mathbf{M_B} = -W_1(L + a) + R_A.L - W_2\left(\frac{L}{2}\right)$

  $\mathbf{M_D} = 0$

Check – $\mathbf{M_D}$ = $\boldsymbol{-W_1(L + 2a) + R_A(L + a) - W_2\left(\frac{L}{2} + a\right) + R_B.a = 0}$

➢ *OVERHANGING BEAM WITH UDL :*

Consider a beam of length L having overhangs at CA & BD, each equal to 'a'. it carries an UDL of w/unit length over the entire length. R_A & R_B are the reactions at the support A & B.

$\Sigma F_Y = 0$ i.e. $R_A + R_B = w(L + 2a)$

By symmetry, $R_A = R_B = \frac{w(L+2a)}{2}$

- **SHEAR FORCE :**

Shear force at C, $F_C = 0$

Shear force at A, $F_{A\,(\text{just left})} = 0 - w.a = -wa$

$$F_{A\,(\text{just right})} = -wa + R_A = -w.a + \frac{w(L+2a)}{2} = -wa + \frac{wL}{2} + wa = \frac{wL}{2}$$

Shear force at E, $F_E = R_A - w(a + \frac{L}{2}) = \frac{w(L+2a)}{2} - w(a + \frac{L}{2}) = 0$

Shear force at B, $F_{B\,(\text{just left})} = R_A - w(L + a) = \frac{w(L+2a)}{2} - w(L + a) = \frac{wL}{2} + wa - wL - wa = \frac{-wL}{2}$

$$F_{B\,(\text{just right})} = \frac{-wL}{2} + R_B = \frac{-wL}{2} + \frac{w(L+2a)}{2} = \frac{-wL}{2} + \frac{wL}{2} + wa = wa$$

Shear force at D, $F_D = R_A + R_B - w(L + 2a) = \frac{w(L+2a)}{2} + \frac{w(L+2a)}{2} - w(L + 2a)$

$$= \frac{wL}{2} + wa + \frac{wL}{2} + wa - wL - 2wa = 0$$

- **BENDING MOMENT :**

$M_C = 0$

$M_A = wa\,(\frac{a}{2}) = wa^2/2$

$M_E = R_A.\frac{L}{2} - w(\frac{L}{2} + a).\frac{1}{2}(\frac{L}{2} + a) = R_A.\frac{L}{2} - \frac{w}{2}(\frac{L}{2} + a)^2$

$M_B = R_A.L - w(L + a)(\frac{L+a}{2}) = R_A.L - \frac{w}{2}(L + a)^2$

$M_D = 0$

THEORY OF SIMPLE BENDING

➢ ***BENDING STRESS :***

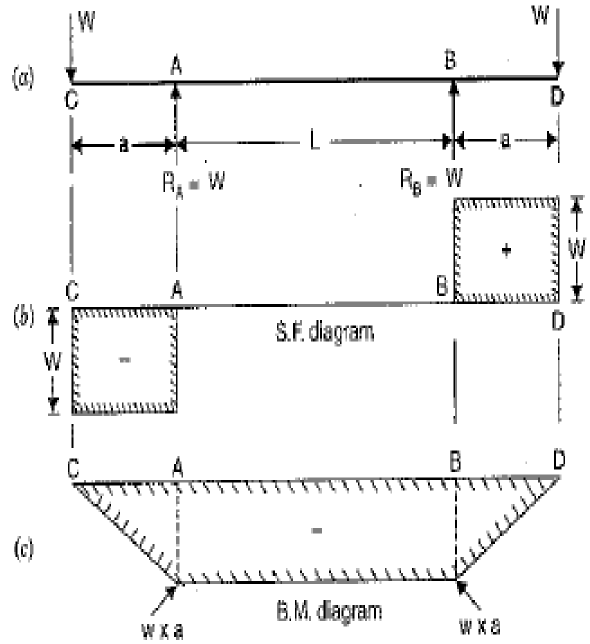
Whenever a beam or cantilever is subjected to external load, the beam or cantilever has a tendency to be bent. Internal stresses are developed in the material of beam to resist this bending.

The resistance offered by internal stresses (developed in beam due to external load) to bending is called **'bending stress'**. Bending stress is tensile & compressive both.

➢ ***PURE BENDING or SIMPLE BENDING :***

The portion of beam free from shear force but subjected to constant bending moment is called 'pure bending'.

From the shear force & bending moment diagram, we find that the portion AB is free from shear force & the bending moment (= W . a) is constant. This portion AB is known as pure bending or simple bending.



➢ ***ASSUMPTIONS ON THE THEORY OF SIMPLE BENDING :***

The following assumptions are made on the theory of simple bending –

- The material of beam is homogeneous (the material is of same kind throughout) & isotropic (the elastic properties in all directions are equal.
- The transverse sections, which are plane before bending remains plane after bending also (i.e. the loads must be perpendicular to the longitudinal axis of the beam)
- The value of Young's Modulus of Elasticity (E) is the same in tension & compression.
- The material of beam obeys Hooke's law & is stressed within its elastic limit.
- Each layer of beam is free to expand or contract independently, of layer above or below it.
- The radius of curvature of beam is very large in comparison to the cross-section dimension of beam.

➢ ***NATURE OF BENDING STRESS :***

If a beam is loaded, it will bend as in fig., whatever may be the nature of load & number of loads. Due to bending of beam, its upper layers are compressed & the lower layers are stretched. Therefore, longitudinal compressive stresses are induced in the upper layers & longitudinal tensile stresses are induced in the lower layers. These stresses are bending stress.

- But in case of cantilevers, reverse will happen i.e. tensile stresses will be induced in the upper layers & compressive stresses will be induced in the lower layers.

➢ ***NEUTRAL LAYER :***

In a beam or cantilever, there is one layer which retains its original length even after bending. So in this layer neither tensile nor compressive stress is set p. this layer is called neutral layer.

➢ ***NEUTRAL AXIS :***

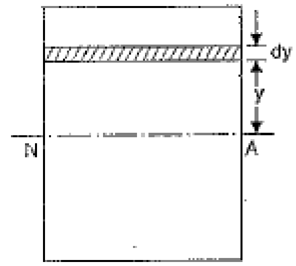
Neutral axis is the line of intersection of neutral layer with any normal section of the beam.

If a cutting plane YY is passed through the length of the beam, the line N.A becomes the line of intersection of the neutral layer & normal section of beam. Therefore, N.A is the neutral axis.

Since no bending stress is setup in the neutral layer, therefore, no bending stress is setup in neutral axis. The stresses are proportional to the distance from N.A.

➢ *POSITION OF NEUTRAL AXIS :*

Neutral axis must contain the centroid of the cross-sectional area or in other words, the neutral axis is a centroidal axis. Thus to locate the neutral axis of a section, we have to find out the centroid of the section.



➢ *BENDING EQUATION :*

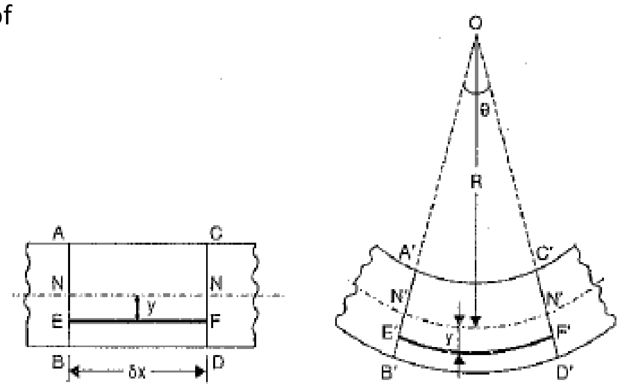
Let us consider any two normal sections AB & CD of a small distance δx apart (i.e. AC = BD = δx)

Let AB & CD intersect the neutral layer at M & N respectively.

Let us consider a layer EF, at any distance 'y' from the neutral layer MN.

After bending of the beam, A'B', C'D', M'N' & E'F' represents the final positions of AB, CD, MN & EF respectively.

When produced A'B' & C'D' intersect each other at O subtending an angle θ radian at O, which is the centre of curvature.



(a) (b)

Since δx is very small, A'B', C'D', M'N' & E'F' may be taken as circular.

Let R = Radius of curvature of neutral layer M'N'.

Now strain in the neutral layer EF due to bending, **e** = $\frac{\mathbf{E'F' - EF}}{EF} = \frac{E'F' - MN}{MN}$ (as EF = MN)

Since MN = M'N' (as neutral layer)

$$\mathbf{e} = \frac{E'F' - M'N'}{M'N'} = \frac{(\mathbf{R} + \mathbf{y}).\theta - \mathbf{R}\,\theta}{R\,\theta} = \frac{R + y - R}{R} = \frac{y}{R} \quad \ldots . \; (1)$$

Let σ_b = Stress set up in layer EF due to bending.

E = Young's modulus of the material of the beam.

Then, $\mathbf{E = \sigma_b/e}$ **or** $\mathbf{e = \sigma_b/E}$ . . . . (2)

Equating equation (1) & (2), we get –

$$\mathbf{y/R = \sigma_b/E} \quad \textbf{or} \quad \mathbf{\sigma_b/y = E/R} \quad \ldots . \; (3)$$

At a distance 'y' , let us consider an elementary strip of very small thickness 'dy'.

σ_b = Bending stress in this strip

let dA = Area of the elementary strip.

Then force developed in this strip = $\boldsymbol{\sigma_b . dA}$

Then elementary moment of resistance due to elementary force is given by –

$$\mathbf{\int dM = \int \sigma_b . dA . y}$$

or $\mathbf{M = \int \sigma_b . dA . y}$

from equation (3), we get –

$$\mathbf{\sigma_b = y . E/R}$$

$$\mathbf{M = \int y . E/R . dA . y} = \frac{E}{R}\mathbf{\int dA . y^2}$$

But, $\mathbf{\int dA . y^2 = I}$ (moment of inertia of whole area about N.A)

$$\mathbf{M} = \frac{E}{R}\mathbf{. I} \quad \textbf{or} \quad \frac{M}{I} = \frac{E}{R}$$

Therefore, $$\frac{M}{I} = \frac{\sigma b}{y} = \frac{E}{R}$$

- ***MOMENT OF RESISTANCE :***

Two equal & unlike parallel forces whose lines of action are not form a couple.

P_c = Resultant compressive force due to compressive stress on one side of N.A.

P_t = Resultant tensile force due to tensile stresses on the other side of it.

$P_c = P_t$

So these two resultant forces form a couple & moment of this couple is equal & opposite to the bending moment at the section where the couple acts.

This moment is called moment of resistance (M.R)

So, moment of resistance of any section of a beam is defined as the moment of the couple formed by the longitudinal internal forces of opposite nature & of equal magnitude, set up at that section on either side of the neutral axis due to bending.

Or, *The moment of the couple developed within the beam section to resist the external bending moment is known as " **Moment of Resistance**".*

- ***DIFFERENCE BETWEEN B.M. & M.R. :***

1. Bending moment develops in a member due to external force, where as Moment of Resistance is developed due to bending moment to resist the same.

2. Bending moment at any section depends upon the magnitude of external loads, type of loading etc.
But M.R. depends upon area of the section, nature of area (i.e. T-section or I-section etc.) & type of material.

3. B.M. may be greater than M.R. & is increased due to increase in external loading. But, M.R. is limited to the value of allowable tensile & compressive stress of the material of member.

- While designing the beam, it is always seen that the maximum B.M at any section doesn't exceed M.R. of that section i.e. **M.R = B.M**

- ***FORMULAE FOR MAXIMUM B.M :***

1. Cantilever with one point load at free end –
Maximum B.M = **W . l**
It occurs at the fixed end.

2. Cantilever with UDL over the entire length –
Maximum B.M = $\mathbf{(\omega . l)\ l/2 = \omega l^2/2}$
It occurs at fixed end.

3. Simply Supported beam with one point load at mid span –
Maximum B.M = **W/2 . l/2 = Wl/4**
It occurs at mid span.

4. Simply Supported beam with an UDL over entire length –
Maximum B.M = $\mathbf{\omega l/2 . l/2 - \omega l/2 . l/4 = \omega l^2/4 - \omega l^2/8 = \omega l^2/8}$
It occurs at mid span.

- ***SECTION MODULUS or MODULUS OF SECTION :***

We have, $M/I = \sigma_b/y$

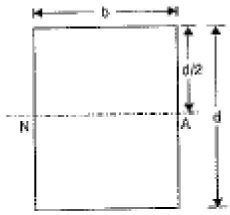
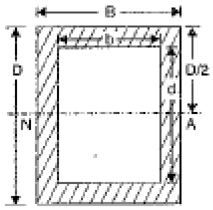
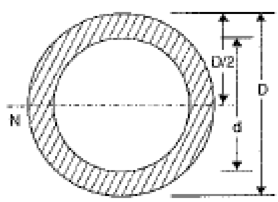
We know that, maximum bending stress ($\sigma_{b(max)}$) occurs at the greatest distance (y_{max}) from neutral axis of the beam.

i.e. $\sigma_{b(max)} = \frac{M}{I} . y_{max}$ or $M = \sigma_{b(max)} . I/Y_{max} = \sigma_{b(max)} . Z$

where, I/y_{max} = Z is called Modulus of section or Section modulus of beam.

i.e. $Z = \frac{M.I. of\ the\ beam\ cross-section\ above\ the\ N.A}{Distance\ of\ extreme\ fibre\ from\ N.A}$

thus, Section Modulus is defined as the ratio of Moment of Inertia of a beam section about the neutral axis to the distance of extreme fibre from the neutral axis,

SECTION	Y_{max}	I	$Z = I/y_{max}$
Rectangular solid 	d/2 b/2	$I_{xx} = \frac{1}{12} bd^3$ $I_{yy} = \frac{1}{12} db^3$	$bd^2/6$ $db^2/6$
Hollow Rectangular 	D/2 B/2	$I_{xx} = \frac{1}{12} (BD^3 - bd^3)$ $I_{yy} = \frac{1}{12} (DB^3 - db^3)$	$\frac{1}{6D} (BD^3 - bd^3)$ $\frac{1}{6B} (DB^3 - db^3)$
Solid Circular	d/2	$I_{xx} = I_{yy} = \frac{\pi}{64} d^4$	$\frac{\pi}{32} d^3$
Hollow Circular 	D/2	$I = \frac{\pi}{64} (D^4 - d^4)$	$\frac{1}{32D} (D^4 - d^4)$
Triangular Section	2h/3	$I = \frac{1}{36} bh^3$	$\frac{1}{24} bh^2$

- ***RELATION BETWEEN MAXIMUM TENSILE STRESS (σ_t) & MAXIMUM COMPRESSIVE STRESS (σ_c) IN ANY SECTION OF BEAM :***

Let us assume an inverted angle section of beam. In case of a beam, maximum compressive stress will be setup in the upmost layer & maximum tensile stress will be setup in the bottom most layer due to bending.

Let, σ_c = max. compressive stress,
σ_t = max. tensile stress,
y_c = distance of top most layer from N.A,
y_t = distance of bottom most layer from N.A.

according to bending equation, we know –

$M/I = \sigma/y$

$M/I = \sigma_c/y_c$ & also **$M/I = \sigma_t/y_t$**

Therefore, **$\sigma_c/y_c = \sigma_t/y_t$** or ***$\sigma_c/\sigma_t = y_c/y_t$***

- ***FLEXURAL RIGIDITY :***

  From bending equation, we know that - $\frac{M}{I} = \frac{E}{R}$

  Where, M = M.R. at any section of beam,
  I = M.I. of section about N.A,
  E = Young's modulus of material of beam,
  R = Radius of curvature of neutral layer.

  The product of I & E is constant for any section of the beam & called flexural rigidity (i.e. = EI)

- ***NATURE OF DISTRIBUTION OF BENDING STRESS :***

It will be proved that the bending stress at any layer of section of a beam varies directly as the distance of layer from N.A (i.e. σ α y)

The bending stress is maximum at a layer where distance from N.A is maximum.

The bending stress gradually decreases as we proceed towards neutral axis. It becomes zero at the neutral axis & then again increases in the reverses direction as the distance of layer from N.A. is increased.

The arrows indicating the magnitude of bending stress at different layers of a section above & below the neutral axis have been gives in opposite directions just to show the difference in nature of stresses in these areas.

- ***FLITCHED BEAM :*** It is that beam which consists two materials. It is used to ensure less weight of material of beam keeping strength constant.
- ***MODULAR RATIO :***

$$m = \frac{Young's\ modulus\ of\ stronger\ material}{Young's\ modulus\ of\ weaker\ material}$$

COMBINED AXIAL & BENDING STRESS

- ***INTRODUCTION :***

When a vertical load acts on a column or pillar through the c.g. of section i.e. acts axially a direct stress (i.e. compressive) occurs uniformly throughout the section at the base.

But if the load doesn't act along axis, then the load is called eccentric. Due to eccentric loading, bending moments develops in the column & consequently bending stress acts at the base section of column.

So, due to eccentric loading, the column base is subjected to both direct stress & bending stress.

- ***DIRECT STRESS :***

Consider a short column is subjected to an axial compressive load.

Let A = cross-sectional area of column,

P = axial compressive load.

The direct stress, $\boldsymbol{\sigma_c = P/A}$

The intensity of stress is uniform everywhere in the base section as in stress diagram.

- ***COMBINED DIRECT STRESS & BENDING STRESS :***

Consider a short column is subjected to an eccentric load,

P = Eccentric point load,

e = eccentricity of load,

A = cross-sectional area of column = **b.d**

Due to eccentric load, now the column is subjected to both direct & bending stress.

Direct stress, $\boldsymbol{\sigma_d = P/A}$ (comp.)

Due to eccentricity of load, the section is subjected to a bending moment, **M = P.e**

Therefore, stress due to B.M ,Bending stress $\boldsymbol{\sigma_b = \pm M/Z}$

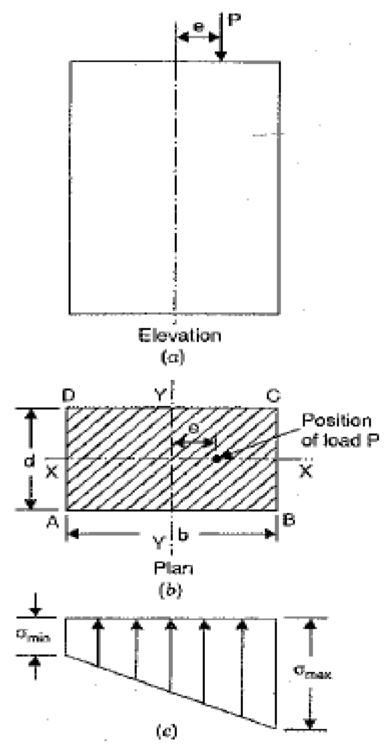
Where, Z = section modulus of section about y-y.

$\boldsymbol{Z = Z_{yy} = (db^3/12)/(b/2) = db^2/6}$

Now, σ_b is positive at the edge B & negative at the edge A. i.e. comp. at B & tensile at A.

Stress at B, $\boldsymbol{\sigma_{max}}$ =Direct stress + bending stress = $\boldsymbol{\sigma_d + \sigma_b = P/A + M/Z}$ (comp.)

at A, $\boldsymbol{\sigma_{min}}$ = Direct stress - bending stress = $\boldsymbol{\sigma_d - \sigma_b = P/A - M/Z}$



- If (+ve), stress is compressive & if (- ve), stress is tensile. As P/A is compressive(+ ve) & M/Z is (- ve).

- ***FOR AN ECCENTRICALLY LOADED MEMBER WITH ECCENTRICITY ABOUT ANY ONE AXIS :***

Resultant stress is normal to the cross-sectional of member = Direct stress $\pm$ Bending stress.

$$= \quad P/A \pm M/Z = \frac{P}{A} + \frac{P\,.\,e}{Z}$$

Where, P = Eccentric load acting on member,

A = Cross-sectional area of member,

M = B.M due to eccentric load,

Z = Section modulus of base section about an axis with respect to which the load is eccentric,

e = **Eccentricity of** load.

- ***COLUMN :*** A long vertical slender member subjected to an axial compressive force is called column.
- ***STRUT :*** *A member is any position other than vertical, subjected to an axial comp. force is called strut.*
- ***CLASSIFICATION OF COLUMNS :***

Columns are classified based on their nature of failure, they are classified as –

a. Short column : In short columns, length is less than 8 times the least lateral dimension (cross-sectional dimension). In this type of column failure is due to bending direct crushing only.

b. Medium column : In medium columns, lengths vary from 8-10 times their least lateral dimension. Failure may occur partly due to crushing & partly due to buckling. Majority of practical columns of this category.

c. ***Long column*** : In large columns, length vary up to 30 times their least lateral dimensions. Failure is only due to buckling (i.e. bending stress). Direct comp. stress is negligible as compared to bending stress.

- ***FAILURE OF COLUMNS :***

➢ A short column fails only by crushing i.e by comp. stress. When the comp. stress is induced in the column due to the given loading is more than that of the comp. yield stress, the short column fails by crushing. The load corresponding to the crushing stress is known as crushing load.

➢ A long column fails at a much lesser stress than a short column, because the long column fails by bending stress. The bending stress develops due to buckling load(which is the less than crushing load). *The limiting load at which bending or buckling just starts in long column is called buckling load or crippling load or critical load or ultimate load*.

- ***ASSUMPTIONS MADE IN THE EULER THEORY :***

  Euler's formula for crippling load is based on the following assumptions –

1. The column is assumed to be made of homogeneous (the material is same kind throughout) & isotropic (the elastic properties in all directions are equal)
2. The column material is perfectly elastic & obeys Hooke's law.
3. Failure of column occurs due to buckling or bending stress alone.
4. The column is initially perfectly straight & axially loaded.
5. The shortening of column due to direct compression is neglected.
6. The length of column is very large in comparison to its lateral dimension (i.e. cross-sectional dimension)

- ***END CONDITIONS OF COLUMN :***

The critical or buckling load of a column depends upon the end conditions of column. The main types of end conditions are –

1. Column with both ends hinged :

L = Effective length or Equivalent length
= The length actually involved in bending
l = Actual length of the column

$$L = l$$

Crippling load, $P = \pi^2 EI/L^2 = \pi^2 EI/l^2$

2. Column with both ends fixed :

$L = l/2$

Crippling load, $P = \pi^2 EI/L^2 = 4\pi^2 EI/l^2$

3. Column with one end fixed & other end hinged :

$$L = l/\sqrt{2}$$

Crippling load, $P = \pi^2 EI/L^2 = 2\pi^2 EI/l^2$

4. ***Column with one end fixed & other end free :***

$$L = 2l$$

Crippling load, $P = \pi^2 EI/L^2 = \pi^2 EI/4l^2$

TORSION

- ***TORSION :***

A member is said to be under pure torsion, when it is subjected to a torque only, without being associated with any bending moment or axial force.

In workshops & factories, rotating shafts are used to transmit energy are subjected to torsion.

- ➢ A beam bends due to bending moment, while a shaft twists due to torsion.
- ➢ When a member is subjected to a moment about its longitudinal axis, then torque is said to be applied & the member is said to be under *torsion.*

- ***ASSUMPTIONS MADE IN TORSION :***

1. The material of the shaft is homogeneous & isotropic.
2. Plane cross-section of a circular shaft remains plane & circular before & after twisting.
3. All diameters of the cross-section of shaft remain straight, with their lengths unchanged, before & after twisting.
4. The twist is uniform along the length of the shaft.
5. Stresses induced in the shaft due to torsion don't exceed the proportional limit.
6. The relative rotation between any two cross-sections of the shaft is proportional to the distance between them.

- ***POLAR MOMENT OF INERTIA :***

The moment of inertia of a plane area with respect to an axis perpendicular to the plane of the area is called the polar moment of inertia of the area with respect to point at which the axis intersects the plane. It is denoted by ***J or*** I_p.

For a ***circular area*** of diameter D, $J = \frac{\pi}{32} D^4$

For a ***hollow circular area*** of outer dia. D & inner dia. d, $J = \frac{\pi}{32} (D^4 - d^4)$

- ***TORSIONAL EQUATION :***

Consider a circular shaft fixed at one end & subjected to a torque at the other end as in fig.

Let, T = Torque in N-mm,
L = Length of the shaft in mm,
R = Radius of the circular shaft in mm,

As a result of torque, every cross-section of the shaft will be subjected to shear stresses.

Let the line CA on the surface of the shaft be deformed to CA' & OA to OA'.

Let angle ACA' = ϕ in degrees
Angle AOA' = θ in radians
τ = Shear stress induced at the surface,
C = Modulus of rigidity (also known as torsional rigidity of material)

We know that, shear strain = **Deformation / unit length**

= AA'/L = tan ϕ = ϕ (ϕ being very small, tan ϕ = ϕ)

We also know that, **arc AA' = R . θ**

AA'/L = R θ /L = ϕ

If τ is the intensity of shear stress in the outer most layer & C is the Modulus of rigidity of shaft, then

$$C = \frac{\tau}{\varphi} \Rightarrow \phi = \frac{\tau}{C} \quad \text{i.e.} \quad \frac{R.\theta}{L} = \frac{\tau}{C} \quad \text{or} \quad \frac{\tau}{R} = \frac{C\,\theta}{L} \; . \; . \; . \; . \quad \textbf{(1)}$$

Now let us consider an elementary area dA in a cross-section of solid circular shaft, which is at a distance of ' r ' from the centre O of the section. dτ be the shear stress induced in elementary area.

Now elementary shear force dF induced in the elementary area dA is given by $dF = d\tau \cdot dA$

Moment of this elementary shear force about the centre O, $dM = dF \cdot r = r\,(d\tau \cdot dA)$

We know that, $\frac{\tau}{R} = \frac{C\,\theta}{L}$

i.e. $d\tau / r = \frac{C\,\theta}{L}$ or $d\tau = \frac{C\,\theta}{L} \cdot r$

$dM = r \cdot d\tau \cdot dA = r \cdot dA \cdot \frac{C\,\theta}{L}\, r = \frac{C\,\theta}{L} r^2 \cdot dA$

Sum total of all such moments developed in the shaft cross-section is known as Torsional moment of resistance.

By integrating the elementary moment dM over the entire cross-section,

$M = \int dM = \int \frac{C\,\theta}{L} r^2 \cdot dA = \frac{C\,\theta}{L} \int r^2\, dA = \frac{C\,\theta}{L} J$

Where, J = Polar moment of inertia of cross-section of shaft = $\int r^2\, dA$

We know that, developed Moment of resistance = applied torque

$M = T$ i.e. $T = \frac{C\,\theta}{L} J$ or $\frac{T}{J} = \frac{C\,\theta}{L}$ . . . . **(2)**

From equation (1) & (2), we get – $\frac{T}{J} = \frac{\tau}{R} = \frac{C\,\theta}{L}$

The above equation is called the Torsional equation for circular shaft.

- ***TORSIONAL MOMENT OF RESISTANCE :***

Whenever a torque is applied in a circular shaft, internal shear stresses are induced in the cross-sections of the shaft. The resultants of these shear stresses form a couple about the longitudinal axis of the shaft. This couple which is numerically equal to the applied external torque is termed as torsional moment of resistance.

$M = \frac{C\,\theta}{L} \cdot J = T$

$\theta = \frac{T\,L}{C\,J}$ where, θ = angle of twist in radians ($1^o = \frac{\pi}{180}$ *radians*)

- ***POWER TRANSMITTED BY A SHAFT :***

Shafts are used to transmit power from a driving motor to a machine tool, from an engine to an axle or from a turbine to an electric motor etc.

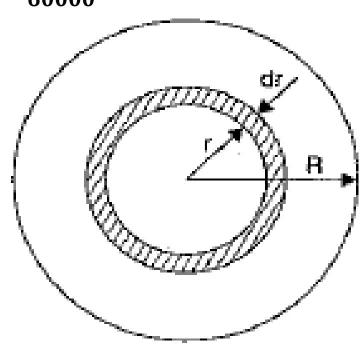
Let, N = No. of revolutions/min. in rpm

T = Average torque in N-m

Power transmitted = Mean torque x angle turned per unit time

$$P = T \cdot \omega = T \cdot \frac{2\pi N}{60} = \frac{2\pi NT}{60} \text{ Nm/sec or Watt } = \frac{2\pi NT}{60000} \text{ kNm/sec or kilo watt (kW)}$$

- ***STRENGTH OF A SOLID SHAFT (Shaft subjected to twisting moment) :***



Strength of a shaft means the maximum torque or power, it can transmit.

Let, R = Radius of the shaft in mm,

T = Shear stress developed in the outer most layer of the shaft in N/mm^2

Consider a shaft subjected to a torque T as in fig.Now let us consider an element of area 'da' of thickness 'dx' at a distance of x from the centre of the shaft.

$da = 2\pi x \cdot dx$ & shear stress at this section = τ_x

Turning force = shear stress x area = $\tau_x \cdot da = \tau \cdot \frac{x}{R} \cdot da = \tau \cdot \frac{x}{R} \cdot 2\pi x \cdot dx = \frac{2\pi\tau}{R} \cdot x^2\, dx$

Turning moment at this element, dT = Turning force x Distance of element from axis of shaft.

$$= \frac{2\pi\tau}{R} \cdot x^2\, dx \cdot x = \frac{2\pi\tau}{R} \cdot x^3\, dx$$

The total torque, the shaft can withstand, may be found out by integrating the above equation **between 0 & R,**

$$T = \int_0^R \frac{2\pi\tau}{R} . x^3\, dx = \frac{2\pi\tau}{R} \int_0^R x^3\, dx = \frac{2\pi\tau}{R} \left[\frac{x^4}{4}\right]_0^R = \frac{\pi}{2}\tau R^3 = \frac{\pi}{2}\tau \left(\frac{D}{2}\right)^3 = \frac{\pi}{16}\tau d^3 \text{ N.mm}$$

$$T = \frac{\pi}{16}\tau d^3 \text{ N.mm}$$

- $T_{max} = \frac{\pi}{16}\tau_{all}\, d^3$
- $T_{max} = \pi\, \theta_a\, d^4\, C/(32L)$

- ***STRENGTH OF A HOLLOW SHAFT :***

It means the maximum torque or power, a hollow shaft can withstand from one pulley to another.

Now consider a hollow circular shaft subjected to some torque.

Let, R = Outer radius of shaft in mm,

r = Inner radius in mm,

τ = Maximum shear stress developed in outer most layer of the shaft material.

Now consider an elementary ring of thickness dx at a distance x from the centre.

Area of the elementary ring, **da = 2 π x dx**

& shear stress at this section, $\tau_x = \tau . \frac{x}{R}$

Therefore **turning force = shear stress x area** $= \tau_x . d_a = \tau . \frac{x}{R} . d_a = \tau . \frac{x}{R} . 2\pi x\, dx = \frac{2\pi\tau}{R} x^2\, dx$

Turning moment of this elementary ring, dT = Turning force x Distance of element from axis of shaft.

$$= \frac{2\pi\tau}{R} x^2\, dx . x = \frac{2\pi\tau}{R} x^3\, dx$$

The total torque, the shaft can withstand, may be determined by integrating the above equation between **r & R**

$$\int_r^R \frac{2\pi\tau}{R} . x^3\, dx = \frac{2\pi\tau}{R} \int_0^R x^3\, dx = \frac{2\pi\tau}{R} \left[\frac{x^4}{4}\right]_r^R = \frac{2\pi\tau}{4R}[R^4 - r^4] = \frac{\pi}{2} . \frac{\tau}{R}[R^4 - r^4]$$

$$= \frac{\pi\tau}{D} . [(D/2)^4 - (d/2)^4] = \frac{\pi}{16} . \tau [(D^4 - d^4)/D] = \frac{\pi}{16} . \tau [D^3 - d^4/D]$$

$$T = \frac{\pi}{16} . \tau . D^3 (1 - d^4/D^4) = \frac{\pi}{16} . \tau . D^3 (1 - k^4) \text{ N.mm}$$ Where, **k = d/D**

- $T_{max} = \pi\, \theta_a\, C\, (D^4 - d^4)/(32L)$

- ***POLAR MODULUS :***

In a circular shaft, the shear stress at any point at a distance ' r ' from the shaft axis is given by –

$\tau_r / r = T / J$ or $\tau = \frac{T\,r}{J}$

maximum shear stress, $\tau = \frac{T\,R}{J} = \frac{T}{J/R}$

The term (J/R) is termed as polar modulus of circular shaft & is denoted by (Z_p)

Hence polar modulus of circular shaft (solid or hollow) is defined as the ratio of its polar moment of inertia to its radius (outermost radius)

- ***TORSIONAL RIGIDITY :***

We know that, angle of twist of a circular shaft under torsion is given as – $\theta = \frac{T\,L}{C\,J}$

we see that, greater the value of JC, the smaller is the angle of twist of the shaft.

The quantity ' JC ' is termed as ' Torsional rigidity of the shaft'.

Hence, Torsional rigidity of any member may be defined as the product of the polar moment of inertia of its cross-section & the modulus of rigidity of the material of the member.

$JC = \frac{T}{\theta/L}$ ------ Torque per angular twist.

- ***MODULUS OF RUPTURE :***

The modulus of rupture of a circular shaft is the maximum fictitious shear stress calculated by the torsional formula by using the experimentally found maximum torque required to rupture the shaft.

Mathematically, $\tau_r = T_u R/J$

Where, τ_r = Modulus of rupture, R = Outer radius of shaft,

T_u = Ultimate torque at failure, J = Polar moment of inertia of shaft.

Although modulus of rupture is a fictitious value of shear stress, it can be used to evaluate the maximum value of torsion that a shaft can carry.

- ***TORSIONAL STIFFNESS (k) :***
  Torsional stiffness, is defined as torque per radian twist. $K = T/\theta = CJ / L$
- ***MAXIMUM SHEAR STRESS :***
  In solid circular shaft, $\tau = 16\,T / (\pi D^3)$
  In hollow circular shaft, $\tau = 16\,T\,D / [\pi (D^4 - d^4)]$
- ***SHAFTS SUBJECTED TO BENDING MOMENT :***
  When the shaft is subjected to a bending moment, then the maximum stress (tensile or comp.) is given by bending equation –

  $\frac{M}{I} = \sigma_b / y$ where, M = Bending moment in N.mm
  $I =$ M.I. of cross-sectional area about the axis of rotation in mm^4
  σ_b = Bending stress, in N/mm^2
  y = Distance from neutral axis to the outermost fibre in mm = d/2

  - ***FOR ROUND SOLID SHAFT :***
    We know that, $I = \frac{\pi}{64} d^4$ & $y = d/2$
    Therefore, the above equation may be written as - $M/(\frac{\pi}{64} d^4) = \sigma_b / (d/2)$
    or $M = \frac{\pi}{32}\ \sigma_b d^3$ ------------ Dia. of solid shaft can be calculated.
  - ***FOR ROUND HOLLOW SHAFT :***
    We know that, $I = \frac{\pi}{64}(d_o^4 - d_i^4) = \frac{\pi}{64} d_o^4 [1 - (d_i/d_o)^4] = \frac{\pi}{64} d_o^4 (1 - k^4)$ where, $k = d_i / d_o$
    Therefore, the bending equation can be written as –
    $M / [\frac{\pi}{64} d_o^4 (1 - k^4)] = \sigma_b / (d_o/2)$
    or $M = \frac{\pi}{32}\ \sigma_b\ d_o^3 (1-k^4)$ ------------ Outside & inside dia. of hollow shaft can be determined.
  - Hollow shafts are usually used in marine works.
  - *When a hollow shaft is to be made equal in strength to a solid shaft, the twisting moment of both shafts must be same.*
    i.e. $T = \frac{\pi}{16} \tau d^3 = \frac{\pi}{16} \tau\ d_o^3 (1 - k^4)$ i.e. $d^3 = d_o^3(1 - k^4)$

- **The twisting moment may be obtained by –**
  $P = \frac{2\pi NT}{60}$ *watt* where, P = Power in watt,
  N = Speed in rpm
  T = Twisting moment in N-mm

  or $P = \frac{2\pi NT}{4500}$ h.p where, P = Power in horse power.

- **In case of belt drive,**
  $T = (T_1 - T_2)R$ where, T_1 = Tension in tight side,
  T_2 = Tension in slack side,
  R = Radius of pulley.

THIN CYLINDRICAL & SPHERICAL SHELL

Cylinders & spheres subjected to fluid pressure are common in engineering practice, such as steam boilers, tanks, chambers of engines, reservoirs etc. These are also known as pressure vessels & shells.

A shell may be termed as thin or thick depending upon the ratio of the thickness of the wall to the diameter of shell.

a. **Thin shell –** if the ratio of the thickness of wall to the internal diameter of a shell is less than $\frac{1}{15}$ (or $\frac{1}{20}$), then it is termed as thin shell. (i.e. t < $\frac{d}{15}$)

b. **Thick shell –** If the ratio of the thickness of wall to the internal diameter of a shell is more than $\frac{1}{15}$ (or $\frac{1}{20}$), then it is termed as thick shell (i.e. t/d > $\frac{d}{15}$)

➢ STRESSES IN THIN CYLINDRICAL SHELL –

Due to internal fluid pressure, thin shells are subjected to circumferential stress or hoop stress (σ_c), longitudinal stress (σ_l) and radial stress (σ_r). Among these stresses, σ_c & σ_l are tensile and σ_r is compressive.

- **Hoop Stress –** It is a tensile stress acting along the circumference of the cylinder. It is denoted by σ_c.
- **Longitudinal Stress –** it is a tensile stress acting along the length of the cylinder. It develops only if cylinder has closed ends. It is denoted by σ_l.
- **Radial Stress –** It is a compressive stress acting along the radius of the cylinder. It is small & is neglected. It is denoted by σ_r.

➢ ASSUMPTIONS IN THIN SHELL –

- The material of shell is homogeneous & isotropic.
- The Hoop stress & Longitudinal stresses are uniform (constant) over the wall thickness.
- Radial stress is small & is neglected.
- Bending of wall is neglected.

➢ DETERMINATION OF HOOP STRESS & LONGITUDINAL STRESS IN A THIN CYLINDRICAL SHELL –

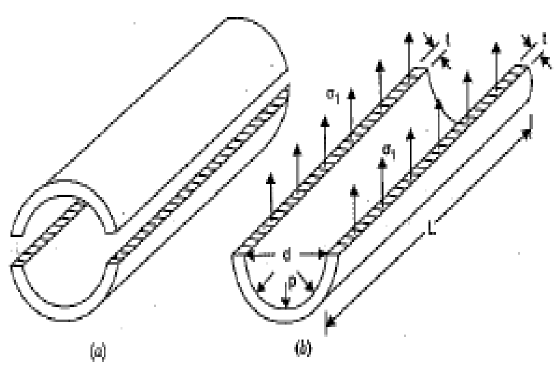
Consider a thin cylindrical shell, subjected to an internal fluid pressure.

Let d = Internal dia. of the shell,
t = Wall thickness of the shell,
p = Intensity of internal fluid pressure,
L = Length of the shell,

σ_c = Hoop stress or circumferential stress,
σ_l = Longitudinal stress,
σ_r = Radial stress is small, hence neglected.

- ***HOOP STRESS :***

Consider half of the cylinder of length L, sectioned through a diametral plane, as in fig.



As the internal pressure in a thin cylinder is low, the radial stresses are small & are neglected.

For equilibrium of forces in the vertical direction,

Resisting force = Resultant vertical pressure force (Bursting Force)

Hoop stress X Resisting area = Intensity of pressure X Projected area

σ_c x Area on which Hoop stress is acting = **p . d.L**

σ_c (2t . L) = p . d.L

or $\sigma_c = \frac{p.d.L}{2t.L} = \frac{pd}{2t}$ i.e. $\boldsymbol{\sigma_c = \frac{pd}{2t}}$

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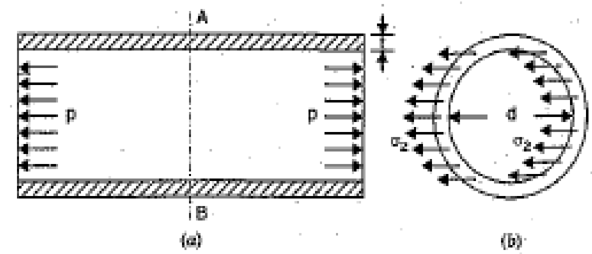
- **LONGITUDINAL STRESS :**

Now consider a section cut by a transverse plane, as in fig.

For equilibrium of forces in the longitudinal direction –

Resisting force = Bursting force (P)

Longitudinal stress X Resisting area = Pressure X Projected area



σ_l x area on which σ_l is acting = $\mathbf{p} \cdot \frac{\pi}{4} \mathbf{d}^2$

$$\sigma_l \cdot \pi\, d\, t = p \cdot \frac{\pi}{4} d^2$$

$$\sigma_l = \frac{p \cdot \frac{\pi}{4} d2}{\pi\, d\, t} = \frac{p\, d}{4\, t} \qquad \text{i.e.} \qquad \sigma_l = \frac{p\, d}{4\, t}$$

- **πdt** is the approximate area. Actually here, d should be taken as the mean diameter.
- From above, **$\sigma_c = 2\, \sigma_l$,** i.e. intensity of longitudinal stress = ½ X Intensity of Hoop stress.
- **σ_c** & **σ_l** both are independent of length of cylinder.

- **MAXIMUM SHEAR STRESS :**

In a cylindrical shell, at any point in the pipe material, there are two principal stresses **σ_c** & **σ_l** .

σ_c acting circumferentially (major) & **σ_l** acting longitudinally & parallel to the axis of the shell (minor).

Both these shells are tensile & act at right angle to each other.

Maximum shear stress, $\tau_{max} = \frac{1}{2}(\sigma_c - \sigma_l) = \frac{1}{2}\left(\frac{p\,d}{2\,t} - \frac{p\,d}{4\,t}\right) = \frac{1}{2}\left(\frac{p\,d}{4\,t}\right)$

$$\tau_{max} = \frac{p\,d}{8\,t}$$

- **HOOP STRAIN:($\varepsilon_c = \delta_d/d$)**

Let, d = Diameter of the shell,

δ_d = change in diameter due to pressure,

p = Intensity of pressure,

E = Young's modulus,

μ = Poisson's ratio

σ_c = Circumferential or Hoop stress

ε_c = Hoop strain.

$$\varepsilon_c = \frac{\textit{change in circumference due t pressure}}{\textit{original circumference}} = \frac{\pi(d+\delta d) - \pi d}{\pi d} = \frac{\delta d}{d}$$

$$\varepsilon_c = \sigma_c/E - \mu.\sigma_l/E = \frac{1}{E}(\sigma_c - \mu\, \sigma_l) = \frac{1}{E}\left(\frac{p\,d}{2\,t} - \mu\,\frac{p\,d}{4\,t}\right) = \frac{p\,d}{2\,t\,E}\left(1 - \frac{\mu}{2}\right)$$

$$\varepsilon_c = \frac{\delta d}{d} = \frac{p\,d}{2\,t\,E}\left(1 - \frac{\mu}{2}\right)$$

- **LONGITUDINAL STRAIN: ($\varepsilon_l = \delta l/l$)**

Longitudinal strain, $\varepsilon_l = \frac{\textit{change in length}}{\textit{original length}} = \delta l/l$

But also, $\varepsilon_l = \sigma_l/E - \mu.\,\sigma_c/E = \frac{1}{E}(\sigma_l - \mu\,\sigma_c) = \frac{1}{E}\left(\frac{p\,d}{4\,t} - \mu\,\frac{p\,d}{2\,t}\right) = \frac{p\,d}{2\,t\,E}\left(\frac{1}{2} - \mu\right)$

Where, l = length of the shell,

δl = change in length.

- **VOLUMETRIC STRAIN: ($\varepsilon_V = \delta V/V$)**

Let, V = Original volume of cylinder = $\frac{\pi}{4} d^2 l$

δV = Change in volume

V + δV = Final volume of shell due to internal pressure = $\frac{\pi}{4}(d + \delta d)^2(l + \delta l)$

$$\delta V = (V + \delta V) - V = \frac{\pi}{4}(d + \delta d)^2(l + \delta l) - \frac{\pi}{4}d^2 l = \frac{\pi}{4}(d^2 + 2\,d\,\delta d + \delta d^2)(l + \delta l) - \frac{\pi}{4}d^2 l$$

$$= \frac{\pi}{4}(d^2 l + 2\,d\,\delta d.l + \delta d^2.l + d^2.\,\delta l + 2\,d\,\delta d.\,\delta l + \delta d^2.\,\delta l) - \frac{\pi}{4}d^2 l$$

$$= \frac{\pi}{4}(d^2 l + 2\,d\,\delta d.l + d^2.\,\delta l) - \frac{\pi}{4}d^2 l \qquad \text{(Neglecting other small quantities)}$$

$$= \frac{\pi}{4}(d^2.\,\delta l + 2\,d\,\delta d.l)$$

Vol. strain, $\varepsilon_V = \frac{\textit{Change in volume}}{\textit{Original volume}}$ *or* $\frac{\delta V}{V} = \frac{\frac{\pi}{4}(d2.\delta l + 2\,d\,\delta d.l)}{\frac{\pi}{4}d2l} = (d^2\delta l)/(d^2 l) + (2\,d\,l\,\delta d)/(d^2 l) = \frac{\delta l}{l} + \frac{2\,\delta d}{d}$

$$\varepsilon_V = \varepsilon_l + 2\,\varepsilon_c = \frac{p\,d}{2\,t\,E}\left(1 - \frac{\mu}{2}\right) + 2\,.\,\frac{p\,d}{2\,t\,E}\left(1 - \frac{\mu}{2}\right) = \frac{p\,d}{2\,t\,E}\left(\frac{1}{2} - \mu + 2 - \mu\right) = \frac{p\,d}{2\,t\,E}(2.5 - 2\mu)$$

- **BUILT-UP CYLINDRICAL SHELLS –**

In actual practice, cylindrical shells of large diameters such as boiler shells etc. are not seamless (without joints), but instead are built-up by longitudinal & circumferential joints.

The longitudinal joints reduce the resisting strength of the shell plate against bursting, and the circumferential joints reduce the resisting strength of the shell plate against due to pressure on end plates.

Let, η_l = Efficiency of longitudinal joint and η_C = Efficiency of circumferential joint.

Then, Bursting force = Resisting strength

i.e. $\mathbf{p \cdot d \cdot l = 2 \cdot l \cdot t \cdot \sigma_C \cdot \eta_l}$ **or** $\mathbf{\sigma_C = (p\,d)/(2\,t\,\eta_l)}$

Similarly, pressure on ends = Resisting force

$\mathbf{P \cdot \frac{\pi}{4} d^2 = \pi\, d\, l\, \sigma_l\, \eta_C}$ **or** $\mathbf{\sigma_l = (p\,d)/(4\,t\,\eta_C)}$

- If the efficiency of joint of shell is considered, i.e. ($\eta_l = \eta_C$) then $\sigma_C = \frac{p\,d}{2\,t\,\eta}$ & $\sigma_l = \frac{p\,d}{4t\,\eta}$

where, η = Efficiency of joint.

- **Changes in length of cylindrical shell due to internal pressure –**

Let, δl = Change in length due to internal pressure & l = length of the shell.

We know that, longitudinal strain, $\varepsilon_l = \frac{\delta l}{l}$

i.e. $\delta l = \varepsilon_l \cdot l = \frac{p\,d}{2\,t\,E}\left(\frac{1}{2} - \mu\right) \cdot l = \frac{p\,d.l}{2\,t\,E}\left(\frac{1}{2} - \mu\right)$ **or** $\delta l = \frac{p\,d.l}{2\,t\,E}\left(\frac{1}{2} - \mu\right)$

- **Changes in diameter of cylindrical shell due to internal pressure –**

Let, δd = Change in diameter due to internal pressure & d = diameter of the shell.

We know that, Hoop strain, $\varepsilon_C = \frac{\delta d}{d}$

i.e. $\delta d = \varepsilon_C \cdot d = \frac{p\,d}{2\,t\,E}\left(1 - \frac{\mu}{2}\right) \cdot d = \frac{p\,d2}{2\,t\,E}\left(1 - \frac{\mu}{2}\right)$ **or** $\delta d = \frac{p\,d2}{2\,t\,E}\left(1 - \frac{\mu}{2}\right)$

- **Changes in Volume of cylindrical shell due to internal pressure –**

Let, δV = Change in Volume due to internal pressure & V = Volume of the shell.

We know that, Volumetric strain, $\varepsilon_V = \frac{\delta V}{V}$

i.e. $\delta V = \varepsilon_V \cdot V = \frac{p\,d}{2\,t\,E}(2.5 - 2\mu) \cdot V = = \frac{p\,d\,V}{2\,t\,E}(2.5 - 2\mu)$ **or** $\delta V = \frac{p\,d\,V}{2\,t\,E}(2.5 - 2\mu)$

- If thickness (t) of thin cylinder is to be determined, then σ_C is used.

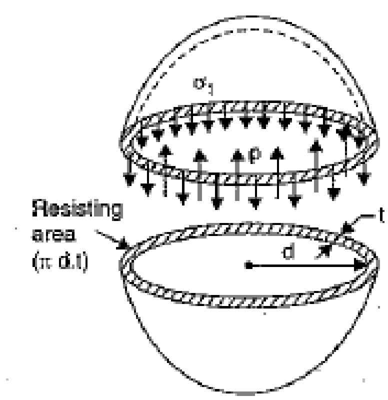
- **THIN SPHERICAL SHELL :**

Consider a thin spherical shell subjected to an internal pressure.

Let, d = Internal diameter of the shell,
t = thickness of the shell
p = Internal fluid pressure.

Hoop stress induced in the shell = σ_C

Considering the equilibrium of vertical forces of hemisphere as in fig.



Resisting force = Bursting force

σ x resisting area = p x projected area

$\sigma \times (\pi d\, t) = p\left(\frac{\pi}{4} d^2\right)$

$\sigma = \left(p \cdot \frac{\pi}{4} d^2\right)/(\pi d\, t) = \frac{p\,d}{4\,t}$

if **η** is the efficiency of the joint, $\sigma = \frac{p\,d}{4\,t\,\eta}$

- In case of spherical shell, 2 principal stresses σ_C & σ_l (along YY axis) at any point are equal & like i.e. $\sigma_C = \sigma_l = \frac{p\,d}{4\,t}$. So no shear stress will be developed at any point in the spherical shell.

- **STRAIN IN SPHERICAL SHELL :**

Strain in any direction, $\varepsilon = \frac{\delta d}{d} = \frac{1}{E}(\sigma_C - \sigma_l.\mu) = \frac{p\,d}{4\,t\,E} - \frac{p\,d\,\mu}{4\,t\,E} = \frac{p\,d}{4\,t\,E}(1 - \mu)$

- **VOLUMETRIC STRAIN IN SPHERICAL SHELL :**

Let, V = Initial volume of the shell = $\frac{\pi}{6}d^3$

δV = Change in volume of shell due to internal pressure.

V + δV = Final volume of shell due to internal pressure = $\frac{\pi}{6}(d + \delta d)^3$

Volumetric strain, $\varepsilon_V = \frac{Change\ in\ volume}{Original\ volume} = \frac{\delta V}{V}$

$$= \frac{(V + \delta V) - V}{V} = \frac{\frac{\pi}{6}(d + \delta d)3 - \frac{\pi}{6}d3}{\frac{\pi}{6}d3}$$

$= (d^3 + 3d^2\delta d - d^3)/d^3$ (By neglecting small quantities)

$$\varepsilon_V = \frac{3\delta d}{d} = 3\varepsilon = \frac{3\,p\,d}{4\,t\,E}(1 - \mu)$$

- Since strain in all direction are same, i.e. $\varepsilon_X = \varepsilon_Y = \varepsilon_Z = \varepsilon$

  Therefore, $\varepsilon_V = \varepsilon_X + \varepsilon_Y + \varepsilon_Z = 3\,\varepsilon$

- **CHANGE IN DIAMETER IN SPHERICAL SPHERE :**

Strain in any direction, $\varepsilon = \frac{\delta d}{d}$

$\delta d = \varepsilon \cdot d$

$$\delta d = \frac{1}{E}(\sigma_C - \sigma_l.\mu).\,d = \left(\frac{p\,d}{4\,t\,E} - \frac{p\,d\,\mu}{4\,t\,E}\right).\,d = \frac{p\,d}{4\,t\,E}(1 - \mu).d = \frac{p\,d2}{4\,t\,E}(1 - \mu).$$

- **CHANGE IN VOLUME IN SPHERICAL SPHERE :**

Volumetric strain, $\varepsilon_V = 3\varepsilon$

$\frac{\delta V}{V} = 3\varepsilon$

Change in volume, $\delta V = 3\,\varepsilon\,V = \frac{3\,p\,d}{4\,t\,E}(1 - \mu).V$

- **Maximum permissible diameter of shell = Minimum of the two values obtained from σ_C & σ_l**
- **Permissible intensity of pressure or Maximum safe air pressure = Minimum of two values obtained from σ_C & σ_l**